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The Third Elliptic Integral and the Ellipsotomic Problem

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IX. *The Third Elliptic Integral and the Ellipsotomic Problem.*

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The Abel Centennial Ceremony, held in Christiania, September, 1902, has directed the attention of mathematicians to the great influence of Abel on modern analysis, and to the history of elliptic functions, and of the foundation by Crelle of the "Journal für die reine und angewandte Mathematik."

Abel's article in the first volume of 'Crelle's Journal,' 1826, "Ueber die Integration der Differential-Formel

$$\frac{\rho\, dx}{\sqrt{\mathrm{R}}} \qquad \text{(A)},$$

wenn R und ρ ganze Functionen sind," is of great importance as indicating the existence of what is now called the *pseudo-elliptic integral;* the present memoir is intended to show the utility of this integral in its application to mechanical theory.

Provided with a list of these integrals, proceeding from the simplest case and extending as far as possible, the student of applied mathematics will be able to effect the complete solution of many interesting mechanical problems now abandoned in an unfinished state; at the same time, the exploration along the simplest lines of progress is effected of the general analytical field, and mathematical research is guided in a path likely to arrive at useful development in the theory of elliptic functions.

The problem of the Division, and thence also of the Transformation, of elliptic functions is solved incidentally; as also the difficulties raised by Legendre in his letter to Abel, of January 16, 1829:—

"Mais de là naît une difficulté sur la division générale des fonctions elliptiques.

"Admettrons-nous cette différence énorme entre les fonctions de 3[me] espèce à paramètre logarithmique et les fonctions à paramètre circulaire, savoir que les premières peuvent s'exprimer par une fonction de deux variables, facilement reductible en table, et que les autres ne le peuvent pas? Il y aurait donc par le fait quatre espèces de fonctions elliptiques au lieu de trois, et la quatrième serait bien plus composée que la troisième. C'est un point qui mérite d'être examiné et mis au clair. Je le recommande à votre investigation et à celle de M. Jacobi."

But a mere change of sign in ABEL'S results is sufficient to pass from the logarithmic to the circular form of the third elliptic integral, and as it is the circular form which is of almost invariable occurrence in dynamical problems, we shall adopt it as our standard form.

Applications of this third elliptic integral are introduced in the course of the memoir to these problems, such as POINSOT'S herpolhode, and the spinning top or gyroscope, the spherical catenary, the velarium, and the elastica under uniform normal pressure.

References to former articles on the same subject in the 'Proceedings of the London Mathematical Society,' vols. 25 and 27, are given in the sequel in the abbreviated form, L.M.S., 25 or 27; frequent reference is made also to KIEPERT'S articles on the theory of elliptic functions in the 'Mathematische Annalen' ('Math. Ann.,' vols. 26, 32, 37).

1. Working then with the third elliptic integral in the circular form, when the elliptic parameter v is a fraction of the imaginary period ω_3, we change the variable in the standard form of WEIERSTRASS,

$$\int_0^u \frac{\frac{1}{2} i\wp' v \, du}{\wp u - \wp v} = i \log e^{u\zeta v} \sqrt{\frac{\sigma(u-v)}{\sigma(u+v)}} \qquad (1),$$

by putting

$$\wp u - \wp v = s - \sigma, \quad \wp'^2 u = \mathrm{S} \qquad (2),$$

$$i\wp' v = \sqrt{-\Sigma}, \quad v = f\omega_3 \qquad (3),$$

where f is a real fraction, so that the integral changes to

$$\int_s^\infty \frac{\frac{1}{2}\sqrt{-\Sigma}}{s-\sigma} \cdot \frac{ds}{\sqrt{\mathrm{S}}} = i \log e^{u\left(\zeta v - \frac{\eta}{\omega} v\right)} \sqrt{\frac{\theta(u-v)}{\theta(u+v)}} \qquad (4)$$

(where the elliptic arguments u and v may be supposed for a moment to be normalised to the Jacobian form), and s is an elliptic function of u which we may denote by $s(u)$, differing from WEIERSTRASS'S $\wp u$ by a constant, so that

$$s'(u) = \wp' u, \quad s''(u) = \wp'' u \ldots,$$

while

$$\sigma = s(v), \quad is'(v) = \sqrt{-\Sigma}.$$

Considering that (HALPHEN, 'Fonctions Elliptiques,' 1, p. 222)

$$\frac{\theta(u-v)}{\theta(u+v)} \qquad (5)$$

is an algebraical function of s when v is an aliquot part of a period, we take

$$i \log \sqrt{\frac{\theta(u+v)}{\theta(u-v)}} = \int_s^\infty \frac{\mathrm{P}(v)(s-\sigma) - \frac{1}{2}\sqrt{-\Sigma}}{s-\sigma} \frac{ds}{\sqrt{\mathrm{S}}} \qquad (6),$$

where
$$iP(v) = \frac{\eta}{\omega} v - \zeta v \tag{7}$$

and denote this integral by I (v), and work with it as our standard form of the elliptic integral of the III. kind.

The function employed by HALPHEN ('F.E.,' 1, p. 230) is now

$$\phi(u, \quad v) = \sqrt{(s - \sigma)} \exp \{ \quad iuP(v) - iI(v)\} \tag{8}$$

$$\phi(u, -v) = \sqrt{(s - \sigma)} \exp \{ -iuP(v) + iI(v)\} \tag{9}$$

and is a Lamé function of the first order, satisfying his differential equation

$$\frac{1}{\phi}\frac{d^2\phi}{du^2} - 2\wp u - \wp v = 0 \tag{10}$$

thence the Lamé functions of higher order may be derived by differentiation, as shown by HERMITE, 'Comptes Rendus,' 1877, and these can be employed in the problems considered by Professor G. H. DARWIN, in the 'Phil. Trans.,' 197, 1901, "Ellipsoidal Harmonic Analysis."

In the Hermite-Jacobi notation we may take

$$\phi(u, v) = \frac{\theta 0 \theta (u - v)}{\theta u \theta v} \exp(uzsv), \quad \text{or} \quad \frac{\theta 0 \mathrm{H}(u - v)}{\theta u \mathrm{H} v} \exp(uznv) \tag{11}$$

2. Next introduce the x and y employed by HALPHEN ('F.E.,' 1, p. 103), which may be connected with the a, b, and p of ABEL's notation ('Œuvres,' II., p. 155) by

$$x = -\frac{16b^3}{p^2}, \quad y = -1 - \frac{4ab}{p} \tag{1}$$

and put

$$s'^2(u) = \mathrm{S} = 4s(s + x)^2 - \{(1 + y)s + xy\}^2 \tag{2}$$

$$s(u) = \wp u + \frac{(1 + y)^2 - 8x}{12} \tag{3}$$

then if we put

$$s - \sigma = s(u) - s(v) = s + x \tag{4}$$

$$\sigma = s(v) = -x, \quad \wp(v) = -\frac{(1 + y)^2 + 4x}{12} \tag{5}$$

$$is'(v) = i\wp'(v) = \sqrt{-\Sigma} = x \tag{6}$$

The multiplicative values

$$s(2v), \quad s(3v), \ldots \; s(nv) \tag{7}$$

$$is'(2v), \quad is'(3v), \ldots \; is'(nv) \tag{8}$$

can now all be expressed rationally in terms of x and y, with the γ functions of HALPHEN ('F.E.,' 1, p. 102; also ABEL, 'Œuvres,' II., p. 159; MONTARD-PONCELOT, 'Applications d'analyse et de géométrie,' t. I., Paris, 1862), by means of the recurring formulas

$$s(mv) - s(nv) = x^{\frac{2}{3}} \frac{\gamma_{m+n}\gamma_{m-n}}{\gamma_m^2 \gamma_n^2}, \quad is'(nv) = x\frac{\gamma_{2n}}{\gamma_n^4} \tag{9}$$

Thus

$$s(v) = -x, \qquad is'(v) = x \tag{10}$$

$$s(2v) = 0, \qquad is'(2v) = xy \tag{11}$$

$$s(3v) = y - x, \qquad is'(3v) = y - x - y^2 \tag{12}$$

$$s(4v) = \frac{x(y-x)}{y^2} - x, \qquad is'(4v) = x\frac{x(y-x-y^2) - (y-x)^2}{y^3} \tag{13}$$

$$s(5v) = \frac{xy(y-x-y^2)}{(y-x)^2} - x, \qquad is'(5v) = x\frac{y^2(xy-x^2-y^3) - x(y-x-y^2)^2}{(y-x)^3} \tag{14}$$

$$s(6v) = \frac{(y-x)(xy-x^2-y^3)}{(y-x-y^2)^2} - x,$$

$$is'(6v) = y\frac{(y-x)^2\{x(y-x-y^2)-(y-x)^2\} - (xy-x^2-y^3)}{(y-x-y^2)^3} \tag{15}$$

$$= x^{\frac{2}{3}}\frac{\gamma_{n+2}\gamma_{n-2}}{\gamma_n{}^2},$$

$$s(nv) = x^{\frac{2}{3}}\frac{\gamma_{n+1}\gamma_{n-1}}{\gamma_n{}^2} - x, \quad is'(nv) = x\frac{\gamma_{2n}}{\gamma_n{}^4} \tag{16}$$

3. For the determination of $\mathrm{P}(nv)$ we have the formula (HALPHEN, 'F.E.,' 1, p. 102)

$$\frac{\sigma(nv)}{(\sigma v)^{n^2}} = \psi_n(v) = (\wp' v)^{\frac{n^2-1}{3}}\gamma_n \tag{1}$$

whence, by logarithmic differentiation,

$$n\mathrm{P}(v) - \mathrm{P}(nv) = i(n\zeta v - \zeta nv)$$

$$= \frac{n^2-1}{3n}\frac{\wp'' v}{i\wp' v} + \frac{1}{n\gamma_n}\frac{d\gamma_n}{dvi}$$

$$= \frac{n^2-1}{3n}(1+y) + \frac{1}{n\gamma_n}\left(\frac{\delta\gamma_n}{\delta x}\frac{dx}{dvi} + \frac{\delta\gamma_n}{\delta y}\frac{dy}{dvi}\right) \tag{2}$$

Now for every homogeneity factor, and as elliptic functions of degree zero,

$$y = \frac{\wp' 2v}{\wp' v} = \frac{s' 2v}{s' v} \tag{3}$$

$$\frac{1}{y}\frac{dy}{dvi} = 2\frac{\wp'' 2v}{i\wp' 2v} - \frac{\wp'' v}{i\wp' v}$$

$$= 2\frac{2x^2 - xy(1+y)}{xy} - \frac{x(1+y)}{x}$$

$$= 4\frac{x}{y} - 3(1+y) \tag{4}$$

and
$$x = \frac{(\wp 2v - \wp v)^3}{\wp'^2 v} \tag{5}$$

$$\begin{aligned}\frac{1}{x}\frac{dx}{dvi} &= -3\,\frac{2i\wp' 2v - i\wp' v}{\wp 2v - \wp v} - 2\,\frac{\wp'' v}{i\wp' v} \\ &= -3\,\frac{2xy - x}{x} - 2(1+y) \\ &= 1 - 8y \end{aligned} \tag{6}$$

so that
$$\begin{aligned} n\mathrm{P}(v) - \mathrm{P}(nv) &= \frac{n^2-1}{3n}(1+y) \\ &+ \frac{1}{n\gamma_n}\left\{x(1-8y)\frac{\delta\gamma_n}{\delta x} + (4x - 3y - 3y^2)\frac{\delta\gamma_n}{\delta y}\right\}\end{aligned} \tag{7}$$

Thus, putting $n = 2, 3, 4, \ldots,$

$$2\mathrm{P}(v) - \mathrm{P}(2v) = \tfrac{1}{2}(1+y) \tag{8}$$

$$3\mathrm{P}(v) - \mathrm{P}(3v) = 1 \tag{9}$$

$$4\mathrm{P}(v) - \mathrm{P}(4v) = \tfrac{1}{2}(1+y) + \frac{x}{y} \tag{10}$$

$$5\mathrm{P}(v) - \mathrm{P}(5v) = 1 + \frac{y^2}{y-x} \tag{11}$$

$$6\mathrm{P}(v) - \mathrm{P}(6v) = \tfrac{1}{2}(1+y) + 1 + \frac{y(y-x)}{y-x-y^2} \tag{12}$$

$$7\mathrm{P}(v) - \mathrm{P}(7v) = 2 + y + \frac{y(y-x)^2}{\gamma_7} \tag{13}$$

$$8\mathrm{P}(v) - \mathrm{P}(8v) = \tfrac{1}{2}(1+y) + 2 + \frac{(y-x)^2(y-x-y^2)}{\gamma_8} \tag{14}$$

$$9\mathrm{P}(v) - \mathrm{P}(9v) = 3 + \frac{x^3(y-x-y^2)^3}{\gamma_9} \tag{15}$$

$$10\mathrm{P}(v) - \mathrm{P}(10v) = \tfrac{1}{2}(1+y) + \frac{\mathrm{N}_{10}}{\gamma_{10}} \tag{16}$$

$$\begin{aligned}\mathrm{N}_{10} = 2x^4(1+y) + x^3(-6y + y^2 + y^3) + x^2(6y^2 - 6y^3 - y^4) \\ + x(-2y^3 + 4y^4 + y^5) - y^6 - y^7,\end{aligned}$$

$$11\mathrm{P}(v) - \mathrm{P}(11v) = \frac{\mathrm{N}_{11}}{\gamma_{11}} \tag{17}$$

$$\begin{aligned}\mathrm{N}_{11} = 3x^5 + 3x^4(-3y + y^2) + x^3(9y^2 + 2y^3 + 8y^4) \\ + x^2(-3y^3 - 15y^4 - 9y^5 + 5y^6) + x(12y^5 + 3y^6 - 2y^7 + y^8) - 2(y^6 + y^7).\end{aligned}$$

4. When v is an aliquot μ^{th} part of the imaginary period $2\omega_3$, so that

$$v = \frac{2\omega_3}{\mu}, \quad \text{or} \quad \frac{2r\omega_3}{\mu} \qquad (1),$$

and μv is congruent to a period, then

$$\mathrm{P}(\mu v) = \infty, \quad \mathrm{P}(mv) + \mathrm{P}(\mu - m)v = 0 \qquad (2),$$

so that $\mu\mathrm{P}(v)$ is given by equation (7), § 3, by putting $n = \mu - 1$, with the additional condition that

$$s(\mu v) = \infty, \quad s(mv) - s(\mu - m)v = 0 \qquad (3),$$

or

$$\gamma_\mu = 0 \qquad (4).$$

Thus

$$\gamma_3 = 0, \quad 3\mathrm{P}(v) = \tfrac{1}{2}(1 + y),$$

$$\gamma_4 = 0, \quad 4\mathrm{P}(v) = 1,$$

$$\gamma_5 = 0, \quad 5\mathrm{P}(v) = \tfrac{1}{2}(3 + x),$$

$$\gamma_6 = 0, \quad 6\mathrm{P}(v) = 2,$$

$$\gamma_7 = 0, \quad 7\mathrm{P}(v) = \tfrac{1}{2}(1 + y) + 1 + \frac{x}{y},$$

$$\gamma_8 = 0, \quad 8\mathrm{P}(v) = 2 + y + \frac{xy}{y - x},$$

$$\gamma_9 = 0, \quad 9\mathrm{P}(v) = \tfrac{1}{2}(1 + y) + 1 + \frac{x}{y} + \frac{y^2}{y - x},$$

$$\gamma_{10} = 0, \quad 10\mathrm{P}(v) = 2 + y + \frac{x}{y} + \frac{y(y - x)}{y - x - y^2},$$

$\gamma_{11} = 0, \quad \mathrm{P}(5v) + \mathrm{P}(6v) = 0$, so that adding (11) and (12), § 3,

$$11\mathrm{P}(v) = 2 + \tfrac{1}{2}(1 + y) + \frac{y^2}{y - x} + \frac{y(y - x)}{y - x - y^2},$$

and, similarly,

$$\gamma_{13} = 0, \quad 13\mathrm{P}(v) = 3 + y + \tfrac{1}{2}(1 + y) + \frac{y(y - x)}{y - x - y^2} + \frac{y(y - x)^2}{\gamma_7},$$

$$\gamma_{15} = 0, \quad 15\mathrm{P}(v) = 4 + y + \tfrac{1}{2}(1 + y) + \frac{y(y - x)^2}{\gamma_7} + \frac{(y - x)^2(y - x - y^2)}{\gamma_8},$$

$$\gamma_{17} = 0, \quad 17\mathrm{P}(v) = 5 + \tfrac{1}{2}(1 + y) + \frac{(y - x)^2(y - x - y^2)}{\gamma_8} + \frac{(y - x - y^2)^3}{x^{-\frac{1}{3}}\gamma_9},$$

$$\gamma_{19} = 0, \quad 19\mathrm{P}(v) = 3 + \tfrac{1}{2}(1 + y) + \frac{(y - x - y^2)^3}{x^{-\frac{1}{3}}\gamma_9} + \frac{\mathrm{N}_{10}}{\gamma_{10}},$$

$$\gamma_{21} = 0, \quad 21\mathrm{P}(v) = \tfrac{1}{2}(1 + y) + \frac{\mathrm{N}_{10}}{\gamma_{10}} + \frac{\mathrm{N}_{11}}{\gamma_{11}}.$$

The ellipsotomic problem of the determination of the division-values (*Theilwerthe*) of the elliptic functions resolves itself thus into a consideration of the curve in x and y given by this rational equation (4), which may thus be called the *ellipsotomic equation*, by analogy with the *cyclotomic equation* for the circular functions; and the problem may be considered solved when x and y can be expressed, rationally or irrationally, in terms of a parameter; according to POINCARÉ ('Bulletin de la Société Mathématique de France,' 1883, 11, p. 112) this can always be effected by uniform functions of an independent variable.

The details have been carried out in the 'Proceedings of the London Mathematical Society' (L.M.S.), 25, for values of μ up to 22 inclusive, omitting 19, which still remains awaiting solution.

When μ is an even number, $4n+2$ or $4n$, $s(\frac{1}{2}\mu v) = s(r\omega_3)$ is a root of $S = 0$, so that the cubic S can be resolved into factors, and we can employ the functions of the Second Stage of LEGENDRE and JACOBI, as required in most dynamical applications.

But when μ is odd, this resolution cannot be effected algebraically, and WEIERSTRASS's functions of the First Stage must be employed.

The ellipsotomic relation (L.M.S., 27, p. 405)

$$\gamma_\mu = 0 \qquad \text{and} \qquad \frac{\gamma(\mu - n)}{\gamma(n)} = \lambda^{\mu - 2n} \tag{5}$$

are equivalent, so that

$$\mu \text{ odd}, \qquad \lambda = \frac{\gamma\,\frac{1}{2}(\mu+1)}{\gamma\,\frac{1}{2}(\mu-1)}, \qquad \lambda^3 = \frac{\gamma\,\frac{1}{2}(\mu+3)}{\gamma\,\frac{1}{2}(\mu-3)}, \ldots \tag{6}$$

$$\mu \text{ even}, \quad \lambda^2 = \frac{\gamma\,\frac{1}{2}(\mu+2)}{\gamma\,\frac{1}{2}(\mu-2)}, \qquad \lambda^4 = \frac{\gamma\,\frac{1}{2}(\mu+4)}{\gamma\,\frac{1}{2}(\mu-4)}, \ldots \tag{7}$$

Writing v for $\dfrac{2\omega_3}{\mu}$, and ω for ω_3,

$$\frac{\sigma n v}{(\sigma v)^{n^2}} = \psi_n(v) = x^{\frac{1}{3}(n^2-1)}\gamma_n \tag{8}$$

(HALPHEN, 'F.E.,' 1, pp. 102, 198); and changing n into $\mu - n$, and dividing,

$$\frac{\sigma(\mu - n)v}{\sigma n v} = [\sigma v x^{\frac{1}{3}} \lambda^{\frac{1}{\mu}}]^{\mu^2 - 2n\mu} \tag{9}$$

But

$$\sigma(\mu - n)v = \sigma(2\omega - nv) = e^{\eta v(\mu - 2n)}\sigma n v \tag{10}$$

so that

$$e^{-\frac{\eta v}{\mu}}\sigma v = x^{-\frac{1}{3}}\lambda^{-\frac{1}{\mu}} \tag{11}$$

and generally

$$e^{-\frac{n^2 \eta v}{\mu}}\sigma n v = x^{-\frac{1}{3}}\lambda^{-\frac{n^2}{\mu}}\gamma_n \tag{12}$$

and this is KIEPERT's function $\tau\left(\dfrac{2n\omega}{\mu}\right)$, defined in 'Math. Ann.,' 32, p. 6.

According to HALPHEN ('Math. Ann.,' 15, p. 359; 'Comptes Rendus,' March, 1879)

$$\gamma_n = \frac{H(nv)\,H(v)^{\frac{1}{3}(n^2-4)}}{H(2v)^{\frac{1}{3}(n^2-1)}} \tag{13}$$

5. The integral employed by ABEL, changed to the circular form, can be written

$$J(v) = \int \frac{Az + K}{\sqrt{Z}}\,dz \tag{1}$$

$$Z = z - (Az^2 + Bz + C)^2 \tag{2}$$

and, as pointed out in the 'Archiv der Mathematik und Physik,' 3. Reihe, I., p. 72, this is exactly in the form required in the problem of LÉVY'S 'Elastica,' with $z = r^2/a^2$.

We reduce it to our standard form in (4), § 1, by putting

$$z = \frac{s(u) - s(3v)}{s(u) - s(v)} = 1 - \frac{y}{s + x} \tag{3}$$

$$Z = z - (Az^2 + Bz + C)^2 = \frac{\frac{1}{4}S}{(s+x)^4} \tag{4}$$

so that

$$(Az^2 + Bz + C)^2 = 1 - \frac{y}{s+x} - \frac{\frac{1}{4}S}{(s+x)^4}$$

$$= \frac{\{2(s+x)^2 - (1+y)(s+x) + x\}^2}{4(s+x)^4} \tag{5}$$

$$Az^2 + Bz + C = 1 - \frac{1+y}{2(s+x)} + \frac{x}{2(s+x)^2} \tag{6}$$

and therefore

$$A = \frac{x}{2y^2}, \qquad B = \frac{-2x + y + y^2}{2y^2}, \qquad C = \frac{x - y + y^2}{2y^2} \tag{7}$$

$$A + B + C = 1 \tag{8}$$

and $z - 1$ is a factor of Z corresponding to $s = \infty$.

More generally, with

$$Mz = \frac{s(u) - s(3nv)}{s(u) - s(nv)} \tag{9}$$

so that $1/M$ is the value of z corresponding to $s = \infty$; then in ABEL'S notation, changing the sign of z and Z to obtain the circular form,

$$Z = pz - (z^2 - az + b)^2 = \frac{\frac{1}{4}QS}{\{s - s(nv)\}^4} \tag{10}$$

$$(z^2 - az + b)^2 = \frac{p}{M}\frac{s - s(3nv)}{s - s(nv)} - \frac{QS}{4\{s - s(nv)\}^4} = \frac{N}{4Mt^4} \tag{11}$$

on putting

$$s - s(nv) = t \tag{12},$$

and

$$\begin{aligned} \mathrm{N} &= 4pt^3\{t - s(3nv) + s(nv)\} \\ &\quad - 4\mathrm{MQ}\{t + s(nv)\}\{t + s(nv) + x\}^2 \\ &\quad + \mathrm{MQ}\{(1+y)t + (1+y)s(nv) + xy\}^2 \\ &= 4pt^4 - 4p\left\{s(3nv) - s(nv) + \frac{\mathrm{MQ}}{p}\right\}t^3 \\ &\quad + \mathrm{MQ}\{(1+y)^2 - 8x - 12s(nv)\}t^2 - 2\mathrm{MQ}s''(nv) - \mathrm{MQ}s'^2(nv) \end{aligned} \tag{13},$$

which is a perfect square in the form

$$\begin{aligned} \mathrm{N} &= \left[2\sqrt{p}t^2 - \sqrt{p}\left\{s(3nv) - s(nv) + \frac{\mathrm{MQ}}{p}\right\}t + \sqrt{\mathrm{MQ}}is'(nv)\right]^2 \\ &= \left\{2\sqrt{p}t^2 - \sqrt{\mathrm{MQ}}\frac{s''(nv)}{is'(nv)} + \sqrt{\mathrm{MQ}}is'(nv)\right\}^2 \end{aligned} \tag{14},$$

implying that

$$s(3nv) - s(nv) + \frac{\mathrm{MQ}}{p} = \sqrt{\frac{\mathrm{MQ}}{p}}\frac{s''(nv)}{is'(nv)} \tag{15},$$

$$\begin{aligned} \frac{\mathrm{MQ}}{p}\{(1+y)^2 - 8x - 12s(nv)\} &= \frac{\mathrm{MQ}}{p}\left\{\frac{s''(nv)}{is'(nv)}\right\}^2 + 4\sqrt{\frac{\mathrm{MQ}}{p}}is'(nv) \\ &= \frac{\mathrm{MQ}}{p}\{(1+y)^2 - 8x - 4s(2nv) - 8s(nv)\} + 4\sqrt{\frac{\mathrm{MQ}}{p}}is'(nv) \end{aligned} \tag{16}.$$

$$\sqrt{\frac{\mathrm{MQ}}{p}} = \frac{is'(nv)}{s(2nv) - s(nv)} \tag{17},$$

which also satisfies (16).

Then

$$\mathrm{M}z = 1 - \frac{s(3nv) - s(nv)}{t} \tag{18},$$

and

$$\begin{aligned} z^2 - az + b &= \frac{2\sqrt{p}t^2 - \sqrt{\mathrm{MQ}}\dfrac{s''(nv)}{is'(nv)}t + \sqrt{\mathrm{MQ}}is'(nv)}{2\sqrt{\mathrm{M}}t^2} \\ &= \sqrt{\frac{p}{\mathrm{M}}} - \tfrac{1}{2}\sqrt{\mathrm{Q}}\frac{s''(nv)}{is'(nv)}\frac{1-\mathrm{M}z}{s(3nv - s(nv)} + \tfrac{1}{2}\sqrt{\mathrm{Q}}is'(nv)\left\{\frac{1-\mathrm{M}z}{s(3nv) - s(nv)}\right\}^2 \end{aligned} \tag{19},$$

requiring the relations

$$1 = \tfrac{1}{2}\sqrt{\mathrm{Q}}is'(nv)\frac{\mathrm{M}^2}{\{s(3nv) - s(nv)\}^2} \tag{20},$$

$$\begin{aligned} a &= \tfrac{1}{2}\mathrm{M}\sqrt{\mathrm{Q}}\left[\frac{s''(nv)}{is'(nv)}\frac{1}{s(3nv) - s(nv)} + \frac{2is'(nv)}{\{s(3nv) - s(nv)\}^2}\right] \\ &= \tfrac{1}{2}\mathrm{M}\sqrt{\mathrm{Q}}\frac{s''(2nv)}{is'(2nv)}\frac{1}{s(3nv) - s(nv)} \end{aligned} \tag{21},$$

$$b = \sqrt{\frac{p}{\text{M}}} - \tfrac{1}{2}\sqrt{\text{Q}}\,\frac{s''(nv)}{is'(nv)}\,\frac{1}{s(3nv) - s(nv)} + \tfrac{1}{2}\sqrt{\text{Q}}\,\frac{is'(nv)}{\{s(3nv) - s(nv)\}^2}$$

$$= \sqrt{\text{Q}}\left\{\frac{s(2nv) - s(nv)}{is'(nv)} - \frac{s(4nv) + s(2nv) - 2s(nv)}{2is'(nv)}\right\}$$

$$= -\sqrt{\text{Q}}\,\frac{s(4nv) - s(2nv)}{2is'(nv)} \qquad (22).$$

Now

$$\text{M}\,dz = \frac{s(3nv) - s(nv)}{t^2}\,dt \qquad (23),$$

$$\sqrt{\text{Z}} = \frac{\sqrt{\text{Q}}\,\sqrt{\text{S}}}{2t^2} \qquad (24),$$

so that

$$\frac{dz}{\sqrt{\text{Z}}} = 2\,\frac{s(3nv) - s(nv)}{\text{M}\sqrt{\text{Q}}}\,\frac{ds}{\sqrt{\text{S}}} \qquad (25),$$

and taking

$$\text{M}\sqrt{\text{Q}} = 2\,\{s(3nv) - s(nv)\} \qquad (26),$$

so that

$$\sqrt{\text{Q}} = 2is'(nv) \qquad (27),$$

$$\text{M} = \frac{s(3nv) - s(nv)}{is'(nv)} = \frac{is'(2nv)}{\{s(2nv) - s(nv)\}^2} \qquad (28),$$

we obtain from (17), (21), (22),

$$p = 4is'(2nv) \qquad (29),$$

$$a = \frac{s''(2nv)}{is'(2nv)} \qquad (30),$$

$$b = -s(4nv) + s(2nv) \qquad (31);$$

and ABEL's integral

$$\int \frac{z-k}{\sqrt{\text{Z}}}\,dz = \int\left\{\frac{is'(nv)}{s(3nv) - s(nv)} - k - \frac{is'(nv)}{t}\right\}\frac{ds}{\sqrt{\text{S}}}$$

$$= \int \frac{2\text{P}(nv)\,t - is'(nv)}{s - s(nv)}\,\frac{ds}{\sqrt{\text{S}}} = 2\text{I}(nv) \qquad (32),$$

on taking

$$k = \frac{is'(nv)}{s(3nv) - s(nv)} - 2\text{P}(nv)$$

$$= \text{P}(2nv) - \text{P}(4nv) = \tfrac{1}{2}a - \text{P}(2nv) \qquad (33).$$

Putting $2nv = w$,

$$p = 4i\wp'w, \qquad a = \frac{\wp''w}{i\wp'w}, \qquad ap = 4\wp''w \qquad (34),$$

so that ABEL'S recurring formula for q_m ('Œuvres,' 2, p. 157),

$$q_m + q_{m-2} = \frac{\frac{1}{2}p^2}{q_{m-1}{}^2} + \frac{ap}{q_{m-1}} \qquad (35),$$

becomes

$$\tfrac{1}{2}q_m + \tfrac{1}{2}q_{m-2} = -\frac{\wp'^2 w}{\frac{1}{4}q_{m-1}{}^2} + \frac{\wp'' w}{\frac{1}{2}q_{m-1}} \qquad (36),$$

and on comparison with the Weierstrassian formula,

$$\wp(m+1)w + \wp(m-1)w - 2\wp w = \frac{\wp'^2 w}{(\wp mw - \wp w)^2} + \frac{\wp'' w}{\wp mw - \wp w} \qquad (37),$$

we infer that

$$\tfrac{1}{2}q_{m-1} = -\wp mw + \wp w \qquad (38).$$

From another formula,

$$\wp(m+1)w - \wp w = \tfrac{1}{2}\frac{\wp'' w}{\wp mw - \wp w} - \tfrac{1}{2}\frac{\wp' w(\wp' mw - \wp' w)}{(\wp mw - \wp w)^2} \qquad (39),$$

ABEL'S relation

$$\frac{c_{m-1}}{p_{m-1}} = \frac{q_m q_{m-1}}{p} = \frac{\{\wp(m+1)w - \wp w\}(\wp mw - \wp w)}{i\wp' w}$$

$$= \tfrac{1}{2}\frac{\wp'' w}{i\wp' w} + \tfrac{1}{2}i\frac{\wp' mw - \wp' w}{\wp mw - \wp w} \qquad (40),$$

and

$$\tfrac{1}{2}i\frac{\wp' mw - \wp' w}{\wp mw - \wp w} = i\zeta(m+1)w - i\zeta mw - i\zeta w$$

$$= \mathrm{P}(m+1)w - \mathrm{P}mw - \mathrm{P}w \qquad (41);$$

so that

$$g_{m-1} = a - \frac{c_{m-1}}{p_{m-1}} = \tfrac{1}{2}a - \mathrm{P}(m+1)w + \mathrm{P}(mw) + \mathrm{P}(w) \qquad (42).$$

Writing μ for ABEL'S $n+2$, his k is given by (2, p. 161)

$$\mu k = (\mu - 1)a - (g_0 + g_1 + \ldots + g_{\mu-3})$$

$$= (\mu-1)a - (\mu-2)\tfrac{1}{2}a + \mathrm{P}(\mu-1)w - (\mu-1)\mathrm{P}w$$

$$= \tfrac{1}{2}\mu a - \mu\mathrm{P}(w) \qquad (43),$$

as before in (33), since

$$\mathrm{P}(\mu-1)w = -\mathrm{P}w \qquad (44).$$

With

$$n = 1, \quad w = 2v,$$

$$\mathrm{M} = \frac{y}{x}, \quad p = 4xy, \quad a = \frac{2x - y - y^2}{y}, \quad b = -x\frac{y - x - y^2}{y^2} \qquad (45),$$

$$k = -2\mathrm{P}(v) + \frac{x}{y} \qquad (46).$$

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But with

$$n = \tfrac{1}{2}(\mu - 1), \quad \wp w = \wp v,$$

$$\mathrm{M} = \frac{x}{\{s\,\frac{1}{2}(\mu - 1)\,v + x\}^2}, \quad p = 4x, \quad a = 1 + y, \quad b = -x \tag{47}$$

$$k = \mathrm{P}(v) - \mathrm{P}(2v) = \mathrm{P}(v) - \mathrm{P}(\mu - 1)\,v = 2\mathrm{P}(v) \tag{48}$$

6. In the simplest case of

$$\mu = 3, \quad v = \tfrac{2}{3}\omega_3, \quad x = 0 \tag{1}$$

the form (6), § 1, is illusory; but we take (L.M.S., 25, p. 210)

$$\mathrm{I}(\tfrac{2}{3}\omega_3) = \int \frac{\frac{1}{6}s + \frac{1}{2}c}{s\sqrt{\{4s^3 - (s+c)^2\}}}\,ds$$

$$= \frac{1}{3}\cos^{-1}\frac{s+c}{2s^{\frac{3}{2}}} = \frac{1}{3}\sin^{-1}\frac{\sqrt{\{4s^3 - (s+c)^2\}}}{2s^{\frac{3}{2}}} \tag{2}$$

and putting $s = t^2$,

$$\mathrm{I} = \frac{2}{3}\cos^{-1}\frac{\sqrt{(2t^3 + t^2 + c)}}{2t^{\frac{3}{2}}} = \frac{2}{3}\sin^{-1}\frac{\sqrt{(2t^3 - t^2 - c)}}{2t^{\frac{3}{2}}} \tag{3}$$

Writing ω for ω_3,

$$12\wp\,\tfrac{2}{3}\,\omega = -1, \quad i\wp'\,\tfrac{2}{3}\,\omega = c, \quad i(\zeta\,\tfrac{2}{3}\,\omega - \tfrac{2}{3}\eta) = \tfrac{1}{6}, \quad e^{-\frac{2\eta\omega}{9}}\sigma\,\tfrac{2}{3}\,\omega = c^{-\frac{1}{3}} \tag{4}$$

In Abel's form ('Œuvres,' 2, p. 163),

$$\mathrm{J} = \int \frac{z + \frac{1}{3}a}{\sqrt{\{(z^2 + az)^2 + pz\}}}\,dz$$

$$= \tfrac{1}{3}\log\left[(z+a)^2 z + \tfrac{1}{2}p + (z+a)\sqrt{\{(z^2+az)^2 + pz\}}\right]$$

$$= \tfrac{2}{3}\log\left[(z+a)\sqrt{z} + \sqrt{\{(z+a)^2 z + p\}}\right] \tag{5}$$

and with

$$z = y^2, \quad p = -q^2,$$

$$\mathrm{J} = \tfrac{4}{3}\log\{\sqrt{(y^3 + ay + q)} + \sqrt{(y^3 + ay - q)}\}$$

$$= \tfrac{4}{3}\,\mathrm{ch}^{-1}\sqrt{\frac{y^3 + ay + q}{2q}} = \tfrac{4}{3}\,\mathrm{sh}^{-1}\sqrt{\frac{y^3 + ay - q}{2q}} \tag{6}$$

Or, in the circular form,

$$\mathrm{J} = \tfrac{4}{3}\cos^{-1}\sqrt{\frac{y^3 + ay + q}{2q}} = \tfrac{4}{3}\sin^{-1}\sqrt{\frac{-y^3 - ay + q}{2q}}$$

$$= -2\int \frac{y^2 + \frac{1}{3}a}{\sqrt{\{-y^2(y^2+a)^2 + q^2\}}}\,dy \tag{7}$$

In the next case (L.M.S., 25, p. 213),

$$\mu = 5, \quad v = \tfrac{2}{5} r\omega, \quad s(5v) = \infty, \quad y = x \tag{1}$$

$$\begin{aligned} \mathrm{I}(v) &= \int \frac{\frac{1}{10}(3+x)(s+x) - \frac{1}{2}x}{(s+x)\sqrt{\mathrm{S}}}\,ds \\ &= \frac{1}{5}\cos^{-1}\frac{(3+x)(s+x)^2 - (1+2x)(s+x) + x}{2(s+x)^{\frac{5}{2}}} \\ &= \frac{1}{5}\sin^{-1}\frac{s+x-1}{2(s+x)^{\frac{5}{2}}}\sqrt{\mathrm{S}} \end{aligned} \tag{2}$$

$$\begin{aligned} \mathrm{I}(2v) &= \int \frac{\frac{1}{10}(1-3x)s - \frac{1}{2}x^2}{s\sqrt{\mathrm{S}}}\,ds \\ &= \frac{1}{5}\cos^{-1}\frac{(1-3x)s^2 - (2x^2 - x^3)s + x^4}{2s^{\frac{5}{2}}} \\ &= \frac{1}{5}\sin^{-1}\frac{s-x^2}{2s^{\frac{5}{2}}}\sqrt{\mathrm{S}} \end{aligned} \tag{3}$$

$$12\wp\tfrac{2}{5}\omega = -1 - 6x - x^2, \quad i\wp'\tfrac{2}{5}\omega = x, \quad \mathrm{P}\tfrac{2}{5}\omega = \frac{3+x}{10}, \quad e^{-\frac{2\eta\omega}{25}}\sigma\tfrac{2}{5}\omega = x^{-\frac{2}{5}} \tag{4}$$

$$12\wp\tfrac{4}{5}\omega = -1 + 6x - x^2, \quad i\wp'\tfrac{4}{5}\omega = x^2, \quad \mathrm{P}\tfrac{4}{5}\omega = \frac{1-3x}{10}, \quad e^{-\frac{8\eta\omega}{25}}\sigma\tfrac{4}{5}\omega = x^{-\frac{3}{5}} \tag{5}$$

and $x^{\frac{1}{5}}$ is the *icosahedron irrationality*, and KIEPERT's $f^{-1} = x$.

In ABEL's form, with $y = x$, Z has the factor $z - x$,

$$\mathrm{Z} = (z-x)\left[-z^3 + (2+x)z^2 - (1-2x)z + x\right] \tag{6}$$

and putting $z - x = \dfrac{x^2}{s}$, ABEL's integral

$$\int \frac{z-k}{\sqrt{\mathrm{Z}}}\,dz = 2\mathrm{I}(2v), \text{ with } k = \frac{1+2x}{5} \tag{7}$$

Putting $s + x = t^2$,

$$\begin{aligned} \mathrm{I}(v) &= \frac{2}{5}\cos^{-1}\frac{(t+1)\sqrt{\{2t^3 - (1-x)t^2 - 2xt + x\}}}{2t^{\frac{5}{2}}} \\ &= \frac{2}{5}\sin^{-1}\frac{(t-1)\sqrt{\{2t^3 + (1-x)t^2 - 2xt - x\}}}{2t^{\frac{5}{2}}} \end{aligned} \tag{8}$$

and the degree of the expressions is halved, with great gain of symmetry.

The degree is halved with greater ease by putting $s = t^2$ in $\mathrm{I}(2v)$, and now

$$\begin{aligned} \mathrm{I}(2v) &= \frac{2}{5}\cos^{-1}\frac{(t-x)\sqrt{\{2t^3 + (1+x)t^2 + 2xt + x^2\}}}{2t^{\frac{5}{2}}} \\ &= \frac{2}{5}\sin^{-1}\frac{(t+x)\sqrt{\{2t^3 - (1+x)t^2 + 2xt - x^2\}}}{2t^{\frac{5}{2}}} \end{aligned} \tag{9}$$

derivable from the preceding $\mathrm{I}(v)$ by writing $\dfrac{t}{x}$ for t, and $-\dfrac{1}{x}$ for x.

8. This suggests that in the general case of $\mu = 2n + 1$ it is simpler to work with

$$\mathrm{I}(2v) = \int \frac{\mathrm{P}(2v)\, s - \frac{1}{2} xy}{s\sqrt{\mathrm{S}}}\, ds \qquad (1),$$

and to put $s = t^2$; and then with

$$\mathrm{T}_1 = 2t^3 + (1 + y)\, t^2 + 2xt + xy \qquad (2),$$

$$\mathrm{T}_2 = 2t^3 - (1 + y)\, t^2 + 2xt - xy \qquad (3),$$

we shall have

$$\mathrm{I}(2v) = \frac{2}{2n+1} \cos^{-1} \frac{t^{n-1} + h_1 t^{n-2} + \ldots + h_{n-1}}{2t^{n+\frac{1}{2}}} \qquad (\sqrt{\mathrm{T}_1}, \text{ or } \sqrt{\mathrm{T}_2})$$

$$= \frac{2}{2n+1} \sin^{-1} \frac{t^{n-1} - h_1 t^{n-2} + \ldots + (-1)^{n-1} h_{n-1}}{2t^{n+\frac{1}{2}}} (\sqrt{\mathrm{T}_2}, \text{ or } \sqrt{\mathrm{T}_1}) \qquad (4),$$

according as n is even or odd; and the results are of one-quarter the degree that would be given by ABEL's method of the periodic continued fraction; and since

$$(t^{n-1} + h_1 t^{n-2} + \ldots)^2 \mathrm{T}_1 + (t^{n-1} - h_1 t^{n-2} + \ldots)^2 \mathrm{T}_2 = 4t^{2n+1} \qquad (5),$$

the determination of $h_1, h_2, \ldots$ can be carried out by a consideration of the *réduites* (HALPHEN, 'F.E.,' 2, p. 576) in preference to continued fractions, once the coefficients of t in T_1 and T_2 have been assigned.*

9. Thus (L.M.S., 25, p. 222) for

$$\mu = 7, \quad v = \frac{2\omega}{7}, \quad \gamma_7 = 0, \quad \text{or} \quad xy - x^2 - y^3 = 0 \qquad (1),$$

is a unicursal C_3, in which

$$x = z(1-z)^2, \quad y = z(1-z) \qquad (2),$$

and

$$\mathrm{T}_1, \mathrm{T}_2 = 2t^3 \pm (1 + z - z^2)\, t^2 + 2z(1-z)^2\, t \pm z^2(1-z)^3 \qquad (3)$$

$$\mathrm{P}(2v) = \frac{3 - 9z + 5z^2}{14} \qquad (4),$$

$$\mathrm{I}(2v) = \int \frac{\mathrm{P}(2v)\, t^2 - \frac{1}{2} z^2 (1-z)^3}{t^2} \frac{dt^2}{\sqrt{\mathrm{T}_1 \mathrm{T}_2}}$$

$$= \frac{2}{7} \cos^{-1} \frac{t^2 + h_1 t + h_2}{2t^{\frac{7}{2}}} \sqrt{\mathrm{T}_2}$$

$$= \frac{2}{7} \sin^{-1} \frac{t^2 - h_1 t + h_2}{2t^{\frac{7}{2}}} \sqrt{\mathrm{T}_1} \qquad (5),$$

$$h_1 = (1-z)^2, \quad h_2 = z(1-z)^3 \qquad (6).$$

Introducing a normalising homogeneity factor M, so that the substitution $\left(z, \frac{1}{1-z}, \frac{z-1}{z}\right)$ should correspond to $(v, 2v, 4v)$,

$$\mathrm{M} = z^{\frac{2}{3}}(1-z)^{\frac{2}{3}} \tag{7}$$

derived from (L.M.S., 27, p. 453),

$$\mathrm{M}^3 = \mathrm{M}_1\mathrm{M}_2\mathrm{M}_4, \quad \mathrm{M}_p = \frac{is'(pv)}{s(2pv) - s(pv)} = x^{\frac{1}{3}} \frac{\gamma_{2p}{}^3}{\gamma_p{}^3\gamma_{3p}} \tag{8}$$

$$\frac{12\wp v}{\mathrm{M}^2} = \frac{-1-6z+9z^2-2z^3-z^4}{z^{\frac{4}{3}}(1-z)^{\frac{4}{3}}}, \quad \frac{i\wp' v}{\mathrm{M}^3} = \frac{1}{z}, \quad \frac{\mathrm{P}(v)}{\mathrm{M}} = \frac{5-z-z^2}{14z^{\frac{2}{3}}(1-z)^{\frac{2}{3}}} \tag{9}$$

$$\frac{12\wp 2v}{\mathrm{M}^2} = \frac{-1+6z-15z^2+10z^3-z^4}{z^{\frac{4}{3}}(1-z)^{\frac{4}{3}}}, \quad \frac{i\wp' 2v}{\mathrm{M}^3} = 1-z, \quad \frac{\mathrm{P}(2v)}{\mathrm{M}} = \frac{3-9z+5z^2}{14z^{\frac{2}{3}}(1-z)^{\frac{2}{3}}} \tag{10}$$

$$\frac{12\wp 4v}{\mathrm{M}^2} = \frac{-1+6z-3z^2-2z^3-z^4}{z^{\frac{4}{3}}(1-z)^{\frac{4}{3}}}, \quad \frac{i\wp' 4v}{\mathrm{M}^3} = \frac{z}{z-1}, \quad \frac{\mathrm{P}(4v)}{\mathrm{M}} = \frac{-1+3z+3z^2}{14z^{\frac{2}{3}}(1-z)^{\frac{2}{3}}} \tag{11}$$

$$\exp\frac{-2\eta\omega}{49}\,\sigma\frac{2\omega}{7} = z^{-\frac{1}{7}}(1-z)^{-\frac{2}{7}}, \quad \exp\frac{-8\eta\omega}{49}\,\sigma\frac{4\omega}{7} = z^{-\frac{2}{7}}(1-z)^{-\frac{4}{7}},$$

$$\exp\frac{-18\eta\omega}{49}\,\sigma\frac{6\omega}{7} = z^{-\frac{4}{7}}(1-z)^{-\frac{1}{7}} \tag{12}$$

In the notation of KLEIN-FRICKE, 'Modulfunctionen,' 2, p. 399,

$$7\mathrm{B}_0{}^2 = -\frac{(1-z+z^2)^2}{z^{\frac{4}{3}}(1-z)^{\frac{4}{3}}} = -4\left\{\frac{\mathrm{P}(v)+\mathrm{P}(2v)+\mathrm{P}(4v)}{\mathrm{M}}\right\}^2$$

$$= 4\frac{\wp v + \wp 2v + \wp 4v}{\mathrm{M}^2} = 4\frac{\mathrm{G}_1}{\mathrm{M}^2} \tag{13}$$

$$\tau = \frac{1-8z+5z+z^2}{z(1-2)} = 5+z+\frac{1}{1-z}+\frac{z-1}{z} \tag{14}$$

10. With (L.M.S., 25, p. 232)

$$\mu = 9, \quad v = \frac{2r\omega}{9} \tag{1}$$

$\gamma_9 = 0$ is a unicursal C_5, in which

$$x = p^2(1-p)(1-p+p^2), \quad y = p^2(1-p) \tag{2}$$

$$\mathrm{T}_1, \mathrm{T}_2 = 2t^3 \pm (1+p^2-p^3)t^2 + 2p^2(1-p)(1-p+p^2)t \pm p^4(1-p)^2(1-p+p) \tag{3}$$

and from the relation

$$\mathrm{P}(4v) + \mathrm{P}(5v) = 0 \tag{4}$$

$$9\mathrm{P}(v) = 1 + \tfrac{1}{2}(1+y) + \frac{x}{y} + \frac{y^2}{y-x} \tag{5}$$

$$18\mathrm{P}(2v) = 4 + 2(1+y) + 4\frac{x}{y} + \frac{4y^2}{y-x} - 9(1+y)$$

$$= 1 + 0 - 3p^2 + 7p^3 \tag{6}$$

$$\mathrm{I}(2v) = \int \frac{\mathrm{P}(2v)\,t^2 - \frac{1}{2}xy}{t^2} \frac{dt^2}{\sqrt{\mathrm{T}_1\mathrm{T}_2}}$$

$$= \frac{2}{9}\cos^{-1}\frac{t^3 + h_1t^2 + h_2t + h_3}{2t^{\frac{3}{2}}}\sqrt{\mathrm{T}_1}$$

$$= \frac{2}{9}\sin^{-1}\frac{t^3 - h_1t^2 + h_2t - h_3}{2t^{\frac{3}{2}}}\sqrt{\mathrm{T}_2}, \quad (7)$$

$$h_1 = -p^2(1-2p), \quad h_2 = -p^4(1-p+p^2), \quad h_3 = p^6(1-p)(1-p+p^2) \quad (8).$$

The substitution $\left(p, \dfrac{p-1}{p}, \dfrac{1}{1-p}\right)$ will correspond to $(v, 2v, 4v)$ with a normalising factor

$$\mathrm{M} = (\mathrm{M}_1\mathrm{M}_2\mathrm{M}_4)^{\frac{1}{3}} = p(1-p) \quad (9),$$

and

$$\frac{12\wp v}{\mathrm{M}^2} = \frac{-1+0-6p^2+10p^3-9p^4+6p^5-p^6}{p^2(1-p)^2}, \quad \frac{i\wp' v}{\mathrm{M}^3} = \frac{1-p+p^2}{p(1-p)^2} \quad (10),$$

$$\frac{12\wp 2v}{\mathrm{M}^2} = \frac{-1+0+6p^2-14p^3+15p^4-6p^5-p^6}{p^2(1-p)^2}, \quad \frac{i\wp' 2v}{\mathrm{M}^3} = \frac{p(1-p+p^2)}{1-p} \quad (11),$$

$$\frac{12\wp 4v}{\mathrm{M}^2} = \frac{-1+12p-30p^2+34p^3-21p^4+6p^5-p^6}{p^2(1-p)^2}, \quad \frac{i\wp' 4v}{\mathrm{M}^3} = \frac{(1-p)(1-p+p^2)}{p^2} \quad (12);$$

while

$$\frac{12\wp 3v}{\mathrm{M}^2} = \frac{-(-1+0+3p^2-p^3)^2}{p^2(1-p)^2} = -\left[\frac{6\mathrm{P}(3v)}{\mathrm{M}}\right]^2, \quad \frac{i\wp' 3v}{\mathrm{M}^3} = 1 \quad (13),$$

unchanged by the substitution.

Also

$$\mathrm{G}_1 = \frac{\wp v + \wp 2v + \wp 3v + \wp 4v}{\mathrm{M}^2} = \frac{(-1+p-p^2)^3}{3p^2(1-p)^2} \quad (14),$$

a quantity required in the transformation of the ninth order; and

$$\frac{\mathrm{P}(v)}{\mathrm{M}} = \frac{5+0+3p^2-p^3}{18p(1-p)} \quad (15),$$

$$\frac{\mathrm{P}(2v)}{\mathrm{M}} = \frac{1+0-3p^2+7p^3}{18p(1-p)} \quad (16),$$

$$\frac{\mathrm{P}(4v)}{\mathrm{M}} = \frac{-7+18p-15p^2+5p^3}{18p(1-p)}$$

$$\frac{\mathrm{P}(3v)}{\mathrm{M}} = \frac{-1+0+3p^2-p^3}{6p(1-p)} = \frac{\eta}{6} = \frac{\xi+3}{6} \quad (17),$$

(Kiepert, 'Math. Ann.,' 32, p. 66).

11. $\mu = 11, \quad v = \frac{2r\omega}{11}, \quad \gamma_{11} = 0$

(L.M.S., 25, p. 241; 'Math. Ann.,' 52, p. 484), leads by the substitutions

$$x = y(1-z), \quad z - y = \frac{z^2}{1+c} \tag{1}$$

to the bicursal C_3

$$z(1-z) + c(c+1)^2 = 0 \tag{2}$$

$$1 - 2z = \sqrt{C}, \quad C = 4c(c+1)^2 + 1 \tag{3}$$

$$x = -\tfrac{1}{2}c(1+c)(1+2c+\sqrt{C}) \tag{4}$$

$$y = \frac{-c-4c^2-2c^3-c\sqrt{C}}{2(1+c)}, \quad 1+y = \frac{2+c-4c^2-2c^3-c\sqrt{C}}{2(1+c)} \tag{5}$$

We now find

$$P(2v) = \frac{6+27c+44c^2+18c^3+(8+13c)\sqrt{C}}{44(1+c)} \tag{6}$$

$$\begin{aligned} I(2v) = \int \frac{P(2v)\,t^2 - \tfrac{1}{2}xy}{t^2}\frac{dt^2}{\sqrt{T_1T_2}} &= \frac{2}{11}\cos^{-1}\frac{t^4+h_1t^3+h_2t^2+h_3t+h_4}{2t^{\frac{11}{2}}}\sqrt{T_2} \\ &= \frac{2}{11}\sin^{-1}\frac{t^4-h_1t^3+h_2t^2-h_3t+h_4}{2t^{\frac{11}{2}}}\sqrt{T_1} \end{aligned} \tag{7}$$

$$h_1 = \frac{2+7c+10c^2+4c^3+(2+3c)\sqrt{C}}{2(1+c)} \tag{8}$$

$$h_2 = -\frac{1+10c+40c^2+82c^3+86c^4+40c^5+6c^6+(1+2c)^2(1+4c+2c^2)\sqrt{C}}{2(1+c)^2} \tag{9}$$

$$h_3 = -\tfrac{1}{2}[1+8c+28c^2+52c^3+50c^4+20c^5+2c^6+(1+2c)(1+4c+6c^2+2c^3)\sqrt{C}] \tag{10}$$

$$h_4 = \frac{c}{2(1+c)}[1+11+54+151+255+254+135+32+2+(1+c)^2 (1+7+19+22)+5)\sqrt{C}] \tag{11}$$

using detached coefficients.

We find also

$$P(rv) = \frac{P_r + Q_r\sqrt{C}}{44(1+c)}, \quad 12\wp rv = \frac{a_r + b_r\sqrt{C}}{2(1+c)^2} \tag{12}$$

where

$$\begin{aligned} P_1 &= 14+19c+0-2c^3, & Q_1 &= 4+c, \\ P_2 &= 6+27c+44c^2+18c^3, & Q_2 &= 8+13c, \\ P_3 &= -2+13c+0-6c^3, & Q_3 &= 12+3c, \\ P_4 &= 12+43c+44c^2+14c^3, & Q_4 &= -6-9c, \\ P_5 &= 4+7c+0-10c^3, & Q_5 &= -2+5c \end{aligned} \tag{13}$$

and the values of a_r and b_r are given in L.M.S., 27, p. 455.

The normalising factor is

$$M = (M_1 M_2 M_3 M_4 M_5)^{\frac{1}{5}} \tag{14}$$

but so far this has not made evident any symmetry of results; it will be noticed that our parameter c here is an elliptic function of (L.M.S., 27, p. 429)

$$u = \int_c^\infty \frac{dc}{\sqrt{C}} \tag{15}$$

which is $\frac{1}{25}$th of a period out of phase with that required to lead to KLEIN's results.

12. $\mu = 13$, $v = \frac{2r\omega}{13}$, $\gamma_{13} = 0$ (L.M.S., 25, p. 251; 'Math. Ann.,' 52, p. 484) by the substitutions

$$x = y(1 - z), \quad z - y = \frac{z^2}{p}, \quad z = c(p - 1) \tag{1}$$

leads to a C_4 with class $p = 2$, in which

$$2p = 1 - c^2 - c^3 + \sqrt{C} \tag{2}$$

$$\begin{aligned} C &= 1 + 4c + 6c^2 + 2c^3 + c^4 + 2c^5 + c^6 \\ &= (1 + 2c - c^2 - c^3)^2 + 4c^2(1 + c)^2 \end{aligned} \tag{3}$$

and we find, using detached coefficients of ascending powers of c,

$$P(2v) = \frac{6 + 12 - 9 - 33 + 4 + 8 - 18 - 11 + (4 + 0 - 15 + 7 + 11)\sqrt{C}}{52(1 + c)^2} \tag{4}$$

$$\begin{aligned} I(2v) &= \int \frac{P(2v)t^2 - \frac{1}{2}xy}{t^2} \frac{dt^2}{\sqrt{T_1 T_2}} \\ &= \frac{2}{13}\cos^{-1}\frac{t^5 + h_1 t^7 + \ldots + h_5}{2t^{\frac{13}{2}}}\sqrt{T_1} \\ &= \frac{2}{13}\sin^{-1}\frac{t^5 - h_1 t^7 + \ldots - h_5}{2t^{\frac{13}{2}}}\sqrt{T_2} \end{aligned} \tag{5}$$

$$h_1 = \frac{1 + 2 - 3 - 9 + 1 + 2 - 5 - 3c^7 + (1 + 0 - 4 + 2 + 3c^4)\sqrt{C}}{2(1 + c)^2} \tag{6}$$

$$h_2 = \frac{A_2 + B_2\sqrt{C}}{2(1 + c)^7},$$

$$\begin{aligned} A_2 &= -1 - 4 - 1 + 16 + 15 - 26 - 28 + 28 + 14 - 38 - 20 + 6 - 5 - 10 - 3c^{14}, \\ B_2 &= -1 - 2 + 4 + 9 - 7 - 15 + 12 + 14 - 7 - 2 + 7 + c'' \end{aligned} \tag{7}$$

$$h_3 = \frac{c^2}{2(1 + c)^6}(A_3 + B_3\sqrt{C}),$$

$$\begin{aligned} A_3 = 1 &+ 6 + 7 - 26 - 64 + 24 + 154 - 6 - 222 + 32 + 266 \\ &+ 10 - 109 + 104 + 143 + 22 + 4 + 32 + 21 + 4c^{19}, \\ B_3 = 1 &+ 4 - 2 - 25 - 10 + 61 + 27 - 97 - 28 \\ &+ 90 - 7 - 82 - 12 + 15 - 15 - 17 - 4c^{16} \end{aligned} \tag{8}$$

$$h_4 = \frac{c^4}{2(1+c)^5}(A_4 + B_4\sqrt{C}),$$

$$A_4 = 1 + 5 + 3 - 21 - 21 + 56 + 46 - 114 - 39 + 158 - 3 - 135$$
$$+ 52 + 91 - 37 - 22 + 35 + 14 - 5 + 2 + 4 + c^{21},$$

$$B_4 = 1 + 3 - 4 - 15 + 14 + 35 - 41 - 37 + 65$$
$$+ 12 - 61 + 8 + 31 - 16 - 14 + 5 + 1 - 3 - c^{18} \quad (9),$$

$$h_5 = -\frac{c^6}{2(1+c)^7}(A_5 + B_5\sqrt{C}),$$

$$A_5 = 1 + 7 + 12 - 22 - 74 + 38 + 223 - 76 - 448 + 205 + 614 - 403 - 551$$
$$+ 555 + 365 - 442 - 68 + 353 + 47 - 99 + 57 + 76 + 7 - 1 + 10 + 6 + c^{26},$$

$$B_5 = 1 + 5 + 1 - 28 - 16 + 91 + 35 - 205 - 2 + 301 - 97 - 290$$
$$+ 169 + 120 - 176 - 79 + 90 + 0 - 54 - 7 + 7 - 5 - 5 - c^{23} \quad (10).$$

By means of an appropriate homogeneity factor M, we can express

$$\frac{52\,P(rv)}{M} = P_r + Q_r\sqrt{C}, \quad \frac{12\wp rv}{M^2} = a_r + b_r\sqrt{C} \quad (11)$$

in such a manner that the substitutions

$$\left(c, -\frac{1+c}{c}, -\frac{1}{1+c}\right) \text{ correspond to } (v, 3v, 9v)$$

and

$$(\sqrt{C}, -\sqrt{C}) \text{ to } (v, 5v)$$

(L.M.S., 25, p. 255; 27, p. 416), and

$$M = (M_1M_2M_3M_4M_5M_6)^{\frac{1}{3}} \text{ or } (M_1M_5)^{\frac{1}{2}} \quad (12).$$

13. $\mu = 15$, $v = \frac{2r\omega_3}{15}$, (L.M.S., 25, p. 258; 'Math. Ann.,' 52, p. 485), $\gamma_{15} = 0$ is reducible to a bicursal C_4.

Changing c in L.M.S., 25, p. 259, into $b - 1$, and normalising by a homogeneity factor

$$M = (M_1M_4)^{\frac{1}{2}} = \left\{\frac{\sqrt{(b^2+b+1)} + \sqrt{(b^2-3b+1)}}{2}\right\}^3 \quad (1),$$

$$I(2v) = \int \frac{Pt^2 - \frac{1}{2}S}{t^2}\frac{dt^2}{\sqrt{T_1T_2}}$$
$$= \frac{2}{15}\cos^{-1}\frac{t^6 - H_1t^5 + \ldots + H_6}{2t^{\frac{15}{2}}}\sqrt{T_2}$$
$$= \frac{2}{15}\sin^{-1}\frac{t^6 + H_1t^5 + \ldots + H_6}{2t^{\frac{15}{2}}}\sqrt{T_1} \quad (2),$$

$$T_1, T_2 = 2t^3 \pm Qt^2 + 2Rt \pm S \quad (3)$$

$$Q = \frac{1+y}{M} =$$

$$\frac{-(b-1)(b^6-2b^5-b^4-b^3+b^2+0+1)-(b+1)(b^4-3b^3+3b^2-b+1)\sqrt{B}}{2b^3\sqrt{(b^2+b+1)}} \quad (4),$$

$$B = (b^2+b+1)(b^2-3b+1) \quad (5),$$

$$R = \frac{x}{M^2} = \frac{(b-1)\{(b^8-4+3+3-2-3+2+2-1)\sqrt{(b^2+b+1)} + (b^8-2-1+1+2-1-2+0+1)\sqrt{(b^2-3b+1)}\}}{2b^4\sqrt{(b^2+b+1)}} \quad (6),$$

$$S = \frac{xy}{M^3} = -\frac{(b-1)^2}{2b^5(b^2+b+1)}$$

$$\{(b^{12}-6+10+1-10-6+12+6-9-5+4+2-1)\sqrt{(b^2+b+1)}$$
$$+(b^{12}-4+2+5+2-8-4+6+5-3-4+0+1)\sqrt{(b^2-3b+1)}\} \quad (7),$$

$$P = \frac{P(2v)}{M} = \frac{(b-1)(13b^6-30-9+3+21+0-7)+(13b^5-30+4+14+0-7)\sqrt{B}}{60b^3\sqrt{(b^2+b+1)}} \quad (8).$$

Differentiating (2) and equating coefficients of t we find

$$H_1 = -\tfrac{1}{4}(15P+Q)$$
$$= \frac{(b-1)(-3b^6+7+2-1-5+0+2)+(-3b^5+7-1-3+0+2)\sqrt{B}}{2b^3\sqrt{(b^2+b+1)}} \quad (9);$$

$$H_2 = -\tfrac{1}{2}R + \tfrac{1}{8}(15P-3Q)H$$
$$= \frac{(b-1)(M_2+N_2\sqrt{B})}{2b^6(b^2+b+1)},$$

$$M_2 = -6b^{13}+33-49+0+8+58-22-39+0+24+4-8-1+1,$$
$$N_2 = -6b^{11}+27-28-13+9+28-7-18+1+7+0-1 \quad (10);$$

$$H_3 = \frac{(b-1)^2(M_3+N_3\sqrt{B})}{2b^9(b^2+b+1)^3},$$

$$M_3 = 4b^{19}-31+80-61-21-54+159+16-113-85$$
$$+97+90-41-64+7+29+2-8-1+1,$$

$$N_3 = +4b^{17}-27+57-23-33-34+84+30-65-46$$
$$+42+40+15-21+3+7+0-1 \quad (11);$$

$$H_4 = \frac{(b-1)^3(M_4+N_4\sqrt{B})}{2b^{18}(b^2+b+1)^2},$$

$$M_4 = 5b^{23}-49+175-247+48+8+468-380-435+93+684$$
$$+48-576-234+350+252-124-150+18+54+3-11-1+1,$$

$$N_4 = 5b^{21}-44+136-145-39+28+303-145-325-19$$
$$+362+69-267-120+131+93-38-41+5+10+0-1 \quad (12);$$

$$H_5 = \frac{(b-1)^4 \{M_5 \surd (b^2+b+1) + N_5 \surd (6^2 - 3b + 1)\}}{2b'' (b^2+b+1)^2},$$

$$M_5 = -b^{25}+13-68+176-209+43+20+305-359-235+298+413$$
$$-308-437+191+386-70-260-3+130+21-44-11+9+2b-1,$$

$$N_5 = -b^{25}+11-46+84-43-23-100+215+59-197-210+231$$
$$+274-143-297+34+230+40-127-54+45+32-7-9b^2+0+1 \quad (13);$$

$$H_6 = -\frac{S}{R} H_5 = \frac{(b-1)^6 \{M_6 \surd (b^2+b+1) + N_6 \surd (b^2-3b+1)\}}{2b^{12} (b^2+b+1)^3},$$

$$M_6 = -b^{29}+15-93+299-494+298+76+387-1151+128+1216$$
$$+370-1762-619+1720+982-1349-1086+773+891$$
$$-288-540+36+232+25-65-13+11+2b-1,$$

$$N_6 = -b^{29}+13-67+165-166-10-56+513-285-588+26+1062$$
$$+152-1165-560+1014-827-594-815+169+562+64$$
$$-266-100+75+49-9-11b^2+0+1 \quad (14).$$

These calculations, as well as for $\mu = 11$ and 13, and their verification, were carried out for me by Mr. J. W. HICKS, of Greenwich Observatory.

Putting

$$\frac{12\wp rv}{M^2} = \frac{a_r + b_r \surd B}{2b^6 (b^2+b+1)} \quad (15),$$

then, since

$$\frac{12\wp 2v}{M^2} = -Q^2 + 8R \quad (16),$$

we find

$$a_2 = -b^{14}+14-43+28+19+22-54-12+30+22-17-8+5+2-1 \quad (17),$$

$$b_2 = -(b-1)(b^{11}-12+19+11-9-19+5+15-1-5+0+1) \quad (18).$$

The substitution $\left(b, \frac{1}{b}\right)$ changes v into $4v$, $2v$ into $8v$, . . ., so that a_8 and b_8 are obtained from a_2 and b_2 by writing the coefficients in reverse order.

$$a_8 = -b^{14}+2+5-8-17+22+30-12-54+22+19+28-43+14-1 \quad (19),$$

$$b_8 = -(b-1)(b^{11}+0-5-1+15+5-19-9+11+19-12+1) \quad (20).$$

Again, since

$$\frac{12\wp v}{M^2} = -Q^2 - 4R \quad (21),$$

we find

$$a_1 = -b^{14}+2+5-8-5-2+18+0-18-2+7+16-7+2-1 \quad (22)$$

$$b_1 = -(b-1)(b^{11}+0-5-1+3+5-7-9-1+7+0+1) \quad (23),$$

and in a_7, b_7 the coefficients run in reverse order.

So also

$$\frac{\mathrm{P}(v)}{\mathrm{M}} = \frac{(b-1)(-b^6+0+3+9+3+0-11)+(-b^5+0+2-8-0-11)\sqrt{\mathrm{B}}}{60b^3\sqrt{(b^2+b+1)}} \qquad (24),$$

and P $(4v)$, P $(8v)$ are obtained from P (v), P $(2v)$ by writing the coefficients in reverse order.

We find also

$$a_3 = -b^{14}+2+5-8-5-2+6+12+6-2-5-8+5+2-1,$$
$$b_3 = -(b^2-1)(b^4-b^3-b^2-b+1)(b^6+0-3b^4+b^3-3b^2+0+1) \qquad (25);$$

$$a_6 = -b^4+2+5+4-29-26+18+60+18-26-29+4+5+2-1,$$
$$b_6 = -(b^2-1)(b^4-b^3-b^2-b+1)(b^6+0-3b^4-11b^3-3b^2+0+1) \qquad (26);$$

$$\frac{\mathrm{P}(3v)}{\mathrm{M}} = \frac{-(b-1)(b^6+0-3+1-3+0+1)-(b+1)(b^4-b^3-b^2-b+1)\sqrt{\mathrm{B}}}{20b^3\sqrt{(b^2+b+1)}} \qquad (27),$$

$$\frac{\mathrm{P}(6v)}{\mathrm{M}} = \frac{(b-1)(3b^6+0-9-17-9+0+3)+3(b+1)(b^4-b^3-b^2-b+1)\sqrt{\mathrm{B}}}{20b^3\sqrt{(b^2+b+1)}} \qquad (28),$$

and these are unchanged by the substitution $\left(b, \frac{1}{b}\right)$.

Also

$$\frac{\mathrm{P}(5v)}{\mathrm{M}} = \frac{-(b^3+1)(b^4-b^3-3b^2-b+1)-(b-1)(b^4+b^3-b^2+b+1)\sqrt{\mathrm{B}}}{12b^3\sqrt{(b^2+b+1)}} \qquad (29)$$

and

$$\frac{12\wp 5v}{\mathrm{M}^2} = -\left[\frac{6\mathrm{P}(5v)}{\mathrm{M}}\right]^2 \qquad (30).$$

With

$$\mathrm{M}' = (\mathrm{M}_1\mathrm{M}_2\mathrm{M}_4\mathrm{M}_8)^{\frac{1}{4}} \qquad (31),$$

$$\mathrm{M}'^2 = \frac{b^3(b-1)}{\sqrt{(b^2+b+1)}}\left[\frac{(b-2)\sqrt{(b^2+b+1)}+b\sqrt{(b^2-3b+1)}}{2}\right]^3 \qquad (32),$$

the expressions for $\frac{12\wp rv}{\mathrm{M}'^2}$ are lowered in degree; for

$$\frac{\mathrm{M}^2}{\mathrm{M}'^2} = \frac{\sqrt{(b^2+b+1)}}{2b^3(b-1)}[(b^5-2-2+2+2-1)\sqrt{(b^2+b+1)}$$
$$-(b^5+0-2-2+0+1)\sqrt{(b^2-3b+1)}] \qquad (33),$$

$$\frac{12\wp rv}{\mathrm{M}'^2} = \frac{(b-1)p_r\sqrt{(b^2+b+1)}+bq_r\sqrt{(b^2-3b+1)}}{2b^3(b-1)\sqrt{(b^2+b+1)}} \qquad (34),$$

$$p_1 = -2b^6 + \ 3 + 3 + 1 - 9 - 9 - 2, \qquad q_1 = 3(-b^5 + 0 - 2 + 2 + 0 + 3) \quad (35),$$

$$p_4 = -2 \ - \ 9 - 9 + 1 + 3 + 3 - 2, \qquad q_4 = 3(3 + 0 + 2 - 2 + 0 - 1) \quad (36),$$

$$p_2 = -2 \ + 15 - 21 + 13 - 9 + 3 - 2, \quad q_2 = 3(3 - 4 + 2 - 2 + 4 - 1) \quad (37),$$

$$p_8 = -2 \ + \ 3 - 9 + 13 - 21 + 15 - 2, \quad q_8 = 3(-1 + 4 - 2 + 2 - 4 + 3) \quad (38),$$

$$p_1 + p_4 + p_2 + p_8 = -4(2b^2 - b + 2)(b^4 - b^3 + 3b^2 - b + 1) \quad (39),$$

$$q_1 + q_4 + q_2 + q_8 = 12(b^5 + 1) \quad (40);$$

$$p_3 = -2 + 3 + 3 - 11 + 3 + 3 - 2, \quad q_3 = 3(b + 1)(-b^4 + 1 + 1 + 1 - 1) \quad (41),$$

$$p_6 = -2 + 3 + 3 + 13 + 3 + 3 - 2, \quad q_6 = 3(b + 1)(-b^4 + 1 + 1 + 1 - 1) \quad (42),$$

so that

$$\wp 6v - \wp 3v = \mathrm{M}'^2 \quad (43);$$

$$p_5 = -2 + 3 + 3 + 1 + 3 + 3 - 2, \quad q_5 = 3(b + 1)(-b^4 + 1 - 3 + 1 - 1) \quad (44),$$

14. The case of $\mu = 15$ can also be derived by a trisection of $\mu = 5$, and so generally when $\mu = 3n$ is any multiple of 3.

For when

$$\mathrm{S} = 4s(s + x)^2 - \{(1 + y)s + xy\}^2 \quad (1)$$

is expressed in the form

$$\mathrm{S} = 4(s + t)^3 - (\mathrm{A}s + \mathrm{B})^2 \quad (2),$$

this implies that

$$s(\tfrac{2}{3}\omega) = -t, \quad is'(\tfrac{2}{3}\omega) = \mathrm{A}t - \mathrm{B} \quad (3);$$

and now

$$\mathrm{A}^2 = 12t + (1 + y)^2 - 8x = -12\wp\tfrac{2}{3}\omega \quad (4),$$

$$2\mathrm{AB} = 12t^2 - 4x^2 + 2xy(1 + y) \quad (5),$$

$$\mathrm{B}^2 = 4t^3 + x^2y^2 \quad (6),$$

so that

$$\{12t + (1 + y)^2 - 8x\}(4t^3 + x^2y^2) - (6t^2 - 2x^2 + xy + xy^2)^2 = 0 \quad (7),$$

reducing to

$$3t^4 + \{(1 + y)^2 - 8x\}t^3 + 3x(2x - y - y^2)t^2 + 3x^2y^2t + x^3(y - x - y^2) = 0 \quad (8),$$

a Jacobian quartic for t, which can be resolved.

For

$$n = 5, \quad y = x,$$

$$3t^4 + (x^2 - 6x + 1)t^3 - 3x^2(x - 1)t^2 + 3x^4t - x^5 = 0 \quad (9),$$

and putting $t = cx$,

$$(c - 1)^3x^2 + 3c^2(c - 1)^2x + c^3 = 0 \quad (10).$$

To identify with the preceding results in § 13, put

$$c = \frac{b^2 + b + 1}{3b}, \quad c - 1 = \frac{(b-1)^2}{3b}, \quad 3c + 1 = \frac{(b+1)^2}{b}, \quad 3c - 4 = \frac{b^2 - 3b + 1}{b} \quad (11),$$

and then the associated octahedron irrationality $o = \frac{1}{2}$.

15. For $\mu = 17$, the quartic for q in terms of c is given in L.M.S., 25, p. 264, obtained by the substitution in $\gamma_{17} = 0$ of

$$x = y(1 - z), \quad y = z\left(1 - \frac{z}{p}\right), \quad z = \frac{c(1+c)}{q-1}, \quad p = \frac{q+c}{q-1} \quad (1).$$

This irreducible quartic is made reducible by putting $q = -c(e+1)$, and with $c = b - 1$ becomes

$$e(e+1)b^4 + e(2e^2 + 2e + 1)b^3 + (e^4 - e^3 - 3e^2 - 3e - 1)b^2$$
$$- e(e+1)(2e^2 + 2e + 1)b + e(e+1)^3 = 0 \quad (2),$$

$$\left\{b - \frac{e+1}{b} + \frac{2e^2 + 2e + 1}{2(e+1)}\right\}^2 = \frac{4e^2 + 9e + 4}{4e(e+1)^2} \quad (3).$$

The alternating function

$$\frac{s(8v) - s(2v)}{s(4v) - s(v)} = \frac{(b-1)(b+e+1)}{eb} = \frac{-\sqrt{e} + \sqrt{(4e^2 + 9e + 4)}}{2(e+1)e^{\frac{3}{2}}} \quad (4),$$

and the division values are associated with elliptic functions of an argument

$$u = \int_e^\infty \frac{de}{\sqrt{(4e^3 + 9e^2 + 4e)}} \quad (5),$$

$$e(u + \omega) = \frac{1}{e}, \quad e(\tfrac{1}{2}\omega) = 1, \quad e(\omega' + \tfrac{1}{2}\omega) = -1, \quad o = \frac{\sqrt{17} - 1}{4} \quad (6),$$

$$f = e(u + \omega' + \tfrac{1}{2}\omega) = -\left\{\frac{-\sqrt{e} + \sqrt{(4e^2 + 9e + 4)}}{2(e+1)}\right\}^2, \quad (e+1)^2(f+1)^2 + ef = 0 \quad (7).$$

By the transformation

$$4e + 9 + \frac{4}{e} = t^2, \quad \frac{4(e+1)^2}{e} = t^2 - 1, \quad \frac{4(e-1)^2}{e} = t^2 - 17 \quad (8),$$

$$u = \int \frac{dt}{\sqrt{(t^2 - 1 \,.\, t^2 - 17)}} \quad (9),$$

and a comparison can be made with the equations of KIEPERT ('Math. Ann.,' 37, p. 386).

16. The next case of $\mu = 19$ presents difficulties not yet surmounted, although it was hoped that the analogy with $\mu = 11$ would give the clue required.

In the case of $\mu = 11$, the substitution of

$$1 - z = (1 + c)\left(1 + \frac{1}{a}\right) \quad (1)$$

in (2), § 11, makes the ellipsotomic relation $\gamma_{11} = 0$ equivalent to

$$a^2c^2 - 2ac - a - c - 1 = 0 \tag{2},$$

an addition equation of the elliptic functions a and c of the argument

$$u = \int_c^\infty \frac{dc}{\sqrt{C}}, \quad C = 1 + 4c(1+c)^2 \tag{3},$$

a and c differing in phase by one-fifth of a period; and the five division values of the arguments $2^r v$, $v = \frac{2}{11}\omega_3$, are derivable from each other when considered as elliptic functions of $u + \frac{2}{5}r\omega$, $r = 0, 1, 2, 3, 4$.

The connexion with KLEIN's parameter τ is made through

$$K = -11\tau, \quad K'^2 = 4K^2(K-11) + (10K+11)^2 = 121\tau'^2 \tag{4},$$

and the quintic transformation

$$H = \frac{1 + 4c + 2c^2 - 5c^3 - 2c^4 + c^5}{c^2(1+c)^2}$$

$$= \frac{x^{11}+1}{x^2(x^2+1)^2(x^3+1)}, \quad \text{if} \quad \frac{1+c}{c} = x + \frac{1}{x}$$

$$H'^2 = \frac{(2 + 8c + 12c^2 + 9c^3 - c^4 - 3c^5 - c^6)^2 C}{c^6(1+c)^6} \tag{5},$$

and then

$$H^2K^2 - 110HK - 121(H+K) + 1331 = 0 \tag{6},$$

an elliptic-function addition equation (L.M.S., 25, p. 244; 27, p. 428).

If analogy is to help us in passing from $\mu = 11$ to 19, the ellipsotomic equation $\gamma_{19} = 0$ should be reducible to the relation

$$H^2K^2 - 152HK - 361(H+K) = 0 \tag{7},$$

where, in terms of FRICK's τ and τ' ('Math. Ann.,' 40),

$$K = -19\tau, \quad K'^2 = 4K^3 + (8K+19)^2 = 361\tau'^2 \tag{8},$$

with the addition of the cubic transformations

$$H = \frac{a^3 - 5a^2 + 2a + 1}{a^2}, \quad H'^2 = \frac{(a^3 - 2a - 2)^2\{4a^3 + (2a+1)^2\}}{a^6} \tag{9},$$

and K, K′ the same functions of b.

This combination of (7) and (9) leads to an equation of the 12th degree between $a = a(u)$ and $b = a\left(u + \frac{2r\omega}{9}\right)$, functions of the elliptic argument

$$u = \int_a^\infty \frac{da}{\sqrt{\{4a^3 + (2a+1)^2\}}} \tag{10}.$$

The nine division values of the argument $2^r v$, $v = \frac{2\omega_3}{19}$ should now be functions of an argument $u + \frac{2r\omega}{9}$, and thence derivable from each other, being grouped in sets of three

$$\left|\begin{array}{ccc} v, & 2^3v, & 2^6v \\ 2v, & 2^4v, & 2^7v \\ 2^2v, & 2^5v, & 2^8v \end{array}\right| \quad \text{or} \quad \left|\begin{array}{ccc} v, & 8v, & 7v \\ 2v, & 3v, & 5v \\ 4v, & 6v, & 9v \end{array}\right| \qquad (11).$$

In passing horizontally in these sets, the substitution connecting $a = a(u)$ and $b = a\left(u + \frac{2r\omega}{3}\right)$ is

$$a^2b^2 - 2ab - a - b = 0 \qquad (12).$$

The additional cubic transformation

$$a = \frac{c^3 + 15c^2 + 57c}{(3c + 19)^2} \qquad (13)$$

leads to a multiplication relation of the 9^{th} order, connecting

$$\text{H} = \text{H}(u) \quad \text{and} \quad c = \text{H}(\tfrac{1}{3}u) \qquad (14).$$

The relation of the 12^{th} order between a and b is of the 6^{th} order in $p = a + b$ and $q = ab$, representing a C_6 in the coordinates p and q.

But so far the various transformations of $\gamma_{19} = 0$, as given in L.M.S., 25, lead to a C_7 of the fifth degree in each variable, and the reason of this is a mystery still.

17. For $\mu = 21$, applying the trisection equation (8) § 14, with the relations of § 9

$$x = z(1 - z)^2, \quad y = z(1 - z) \qquad (1),$$

$$3t^4 + [(1 + z - z^2)^2 - 8z(1 - z)^2]\, t^3$$
$$+ 3z^2(1 - z)^3(1 - 3z + z^2)\, t^2 + 3z^4(1 - z)^6\, t + z^6(1 - z)^7 = 0 \qquad (2).$$

Put

$$t = \frac{z^2(1 - z)^2}{z - w} \qquad (3),$$

$$w^3z^2 - (w^4 - 2w^3 + 3w^2 + 0 - 1)\, z + w(w - 1)^3 = 0 \qquad (4),$$

$$z = \frac{w^4 - 2w^3 + 3w^2 + 0 - 1 + \sqrt{\text{W}}}{2w^3} \qquad (5),$$

$$z - w = \frac{-w^4 - 2w^3 + 3w^2 + 0 - 1 + \sqrt{\text{W}}}{2w^3} \qquad (6),$$

$$\text{W} = w^8 - 8 + 22 - 24 + 11 + 4 - 6 + 0 + 1$$
$$= (w^2 - w + 1)\,[(w^3 - 3w^2 + 0 + 1)^2 - (w^2 - w)(w^3 - 3w^2 + 0 + 1) + (w^2 - w)^2] \qquad (7).$$

The substitution $\left(w, \dfrac{1}{1-w}, \dfrac{w-1}{w}\right)$ gives $\left(z, \dfrac{1}{1-z}, \dfrac{z-1}{z}\right)$, and

$$12\wp 3v = -(1+z-z^2)^2 - 4z(1-z)^2 \tag{8}$$

$$12\wp 6v = -(1+z-z^2)^2 + 8z(1-z)^2 \tag{9}$$

$$s(3v) = -z(1-z)^2, \quad is'(3v) = z(1-z)^2 \tag{10}$$

$$s(6v) = 0, \quad is'(6v) = z^2(1-z)^3 \tag{11}$$

$$s(9v) = z^2(1-z), \quad is'(9v) = z^3(1-z) \tag{12}$$

$$s(7v) = -\frac{z^2(1-z)^2}{z-w}, \quad is'(7v) = \sqrt{\frac{z^4(1-z)^6}{(z-t)^3}\left[4w^2 + (z-w)\left(w + \frac{1}{1-z}\right)^2\right]} \tag{13}$$

18. For $\mu = 23$ the class of the modular equation is 2, so that simple relations cannot be anticipated; but as the class is zero for $\mu = 25$, it is possible that the ellipsotomic equation $\gamma_{25} = 0$ may be susceptible of reduction (L.M.S., 25, p. 275).

19. As mechanical applications of the preceding integrals of the First Stage we may cite the case of LÉVY's "Elastica," mentioned in § 5 and discussed in the 'Math. Ann.,' 52, the Spherical Catenary (L.M.S., 27), and the Velarium surfaces considered in F. KÖTTER's 'Inaugural-Dissertation' (Halle, 1883).

Take an umbrella with straight ribs, and hold its axis vertical, as an illustration of a velarium.

If the gore laid out flat forms a sector of a circle, then it is obvious that for any other angle between the radii formed by the ribs, the edge and its concentric lines form spherical catenary curves, as shown by KÖTTER's equations.

But with triangular gores (KÖTTER, 'Diss.,' p. 38) the projection of the edge on a horizontal plane is given by

$$\theta = \tfrac{1}{2}\int \frac{A\,dt}{t\sqrt{\{(B-\frac{1}{2}t)^2 t - A^2(1-t)\}}} \tag{1}$$

with $t = \left(\dfrac{r}{a}\right)^2$; and this is reducible immediately to our standard form (1), § 5, by putting

$$t = \frac{s}{M^2} \tag{2}$$

$$\begin{aligned} S &= 4s(s+x)^2 - \{(y+1)s + xy\}^2 \\ &= 16M^6T = 4M^2t(M^2t - 2M^2B)^2 + 16M^4A^2(M^2t - M^2) \\ &= 4s(s-2M^2B)^2 + 16M^4A^2(s-M^2) \end{aligned} \tag{3}$$

and equating coefficients,

$$16M^2B = (1 + y)^2 - 8x \quad (4),$$

$$8M^4(A^2 + B^2) = 2x^2 - xy(1 + y) \quad (5),$$

$$4AM^3 = xy \quad (6),$$

so that

$$256M^4A^2 = 64x^2 - 32xy(1 + y) - [(1 + y)^2 - 8x]^2$$
$$= (1 + y)[16x(1 - y) - (1 + y)^3] \quad (7),$$

$$M^2 = \frac{16x^2y^2}{(1 + y)[16x(1 - y) - (1 + y)^3]} \quad (8).$$

Now denoting the elliptic argument by u, where

$$u = \int \frac{ds}{\sqrt{S}} \quad (9),$$

$$\theta + u\mathrm{P}(2v) = \mathrm{I}(2v) \quad (10),$$

and the preceding integrals can be utilised for the construction of solvable cases.

The chief interest is in the purely algebraical case, obtained by putting $\mathrm{P}(2v) = 0$. Thus we find for $\mu = 5$, putting $y = x = \frac{1}{3}$, in (3), § 7,

$$2r^{\frac{5}{2}} \cos \tfrac{5}{2}\theta = (r + a)\sqrt{(2r^3 - 4ar^2 + 6a^2r + 3a^3)} \quad (11),$$

$$2r^{\frac{5}{2}} \sin \tfrac{5}{2}\theta = (r - a)\sqrt{(2r^3 + 4ar^2 + 6a^2r + 3a^3)} \quad (12).$$

20. The expression of the pseudo-elliptic spherical catenary, discussed in L.M.S., 27, p. 127, can be halved in degree by changing to the stereographic projection, with

$$\tan \tfrac{1}{2}\theta = t, \quad z = \cos\theta = \frac{1 - t^2}{1 + t^2} \quad (1);$$

$$Z = (1 - z^2)(h - z)^2 - A^2 = \frac{A^2T}{(1 + t^2)^4} \quad (2),$$

$$T = 4t^2\left(\frac{h - 1}{A} + \frac{h + 1}{A}t^2\right)^2 - (1 + t^2)^4 = T_1T_2 \quad (3),$$

$$T_1, T_2 = 2t\left(\frac{h - 1}{A} + \frac{h + 1}{A}t^2\right) \mp (1 + t^2)^2 \quad (4),$$

$$\psi = \int \frac{A\,dz}{(1 - z^2)\sqrt{Z}} = -\tfrac{1}{2}\int \frac{(1 + t^2)^2\,dt^2}{t^2\sqrt{T_1T_2}} \quad (5).$$

In a pseudo-elliptic case, with

$$\mu = 2n + 1, \quad v = \frac{2\omega_3}{\mu}, \quad M^2 = -\frac{y + 1}{2x}, \quad \frac{A}{M} = y + 1, \quad A^2 - h^2 = 2y + 1 \quad (6),$$

we put

$$\psi + \frac{4\mathrm{P}(v)}{y+1}\int\frac{dt^2}{\sqrt{\mathrm{T}_1\mathrm{T}_2}} = \chi \tag{7}$$

and now the expression for the catenary can be reduced to

$$\begin{aligned}\mathrm{N}(\tan\tfrac{1}{2}\theta e^{\chi i})^{n+\frac{1}{2}} &= (\mathrm{B} + \mathrm{B}_1 t + \mathrm{B}_2 t^2 + \ldots + \mathrm{B}_{2n-1}t^{2n-1})\sqrt{\mathrm{T}_1}\\ &+ i(\mathrm{B} - \mathrm{B}_1 t + \mathrm{B}_2 t^2 - \ldots - \mathrm{B}_{2n-1}t^{2n-1})\sqrt{\mathrm{T}_2}\end{aligned} \tag{8}$$

Thus for

$$\mu = 7, \quad x = z(1-z)^2, \quad y = z(1-z) \tag{9}$$

$$\mathrm{P}(v) = \frac{5 - z - z^2}{14} = 0 \quad \text{for} \quad z = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{21} \tag{10}$$

and we can calculate M, A, h, and the six B's in

$$\begin{aligned}\mathrm{N}(\tan\tfrac{1}{2}\theta e^{\psi i})^{\frac{7}{2}} &= (\mathrm{B} + \mathrm{B}_1 t + \ldots + \mathrm{B}_5 t^5)\sqrt{\mathrm{T}_1}\\ &+ i(\mathrm{B} - \mathrm{B}_1 t + \ldots - \mathrm{B}_5 t^5)\sqrt{\mathrm{T}_2}\end{aligned} \tag{11}$$

this catenary has been drawn stereoscopically by the late Mr. T. I. DEWAR.

21. ABEL's integrals are applicable immediately to the construction in polar co-ordinates r and θ of algebraical orbits or catenaries under a central attraction of the form (HUGO GYLDÉN, 'Kongl. Svenska Vetenskaps-Akademiens Handlingar,' 1879)

$$\mathrm{P} = \frac{\mu}{r^2} + \mu_0 + \mu_1 r \tag{1}$$

with $\mu = 0$ the orbit can be realised by two balls, connected by an elastic thread, whirling round each other in the air; and the addition of a term to P, varying inversely as r^3, merely has the effect of qualifying the angle θ by a factor.

Putting $r = \frac{1}{u}$, and denoting the velocity in the orbit by v and twice the rate of description of area by h,

$$\begin{aligned}v^2 = h^2\left(\frac{du^2}{d\theta^2} + u^2\right) &= \mathrm{H} - 2\int\mathrm{P}\,dr\\ &= 2\frac{\mu}{r} + \mathrm{H} - 2\mu_0 r - \mu_1 r^2\end{aligned} \tag{2}$$

$$u^2\left(\frac{du}{d\theta}\right)^2 = -\mu^4 + 2\mathrm{A}u^3 + \mathrm{B}u^2 - 2\mathrm{C}u - \mathrm{D} = \mathrm{U} \tag{3}$$

$$\mathrm{A} = \frac{\mu}{h^2}, \quad \mathrm{B} = \frac{\mathrm{H}}{h^2}, \quad \mathrm{C} = \frac{\mu_0}{h^2}, \quad \mathrm{D} = \frac{\mu_1}{h^2} \tag{4}$$

$$\theta = \int\frac{u\,du}{\sqrt{\mathrm{U}}} \tag{5}$$

and now put $u = z - k$ to identify with ABEL's results.

Thus for $\mu = 3$ in § 6 (5), putting $u = z + \frac{1}{3}a$, and $a = 3b$,

$$\theta = \frac{2}{3}\cos^{-1}(u+b)\sqrt{\frac{u-b}{p}} = \frac{2}{3}\sin^{-1}\sqrt{\left\{1 - \frac{(u+b)^2(u-b)}{p}\right\}} \tag{6}$$

$$\mathrm{A} = -b, \quad \mathrm{B} = 3b^2, \quad 2\mathrm{C} = -4b^3 - p, \quad \mathrm{D} = b(4b^3 + p) \tag{7}$$

For instance, $b = 0$ makes $\mu = 0$, $\mathrm{H} = 0$, $\mu_1 = 0$, and the orbit is

$$r^{\frac{3}{2}}\cos\tfrac{3}{2}\theta = c^{\frac{3}{2}} \tag{8}$$

described under a constant central force.

For $\mu = 5$ in (7), § 7, putting

$$w = 5z - 1 - 2x \tag{9}$$

$$\theta = \frac{2}{5}\cos^{-1}\frac{w^2 - (8+x)w + 2(8+x)(1-3x)}{50x\sqrt{5}}\sqrt{(w+1-3x)}$$

$$= \frac{2}{5}\sin^{-1}\frac{w-4-3x}{50x\sqrt{5}}\sqrt{\{-w^3 + (7-x)w^2 + 8(-1+11x+x^2)w}$$
$$+ 4(4+3x)(-1+11x+x^2)\} \tag{10}$$

and to make $\mu = 0$, put $x = -3$, so that with $w = 5u$,

$$\theta = \frac{2}{5}\cos^{-1}\frac{u^2 - u + 4}{6}\sqrt{(u+2)}$$

$$= \frac{2}{5}\sin^{-1}\frac{u+1}{6}\sqrt{(-u^3 + 2u^2 - 8u + 4)} \tag{11}$$

an orbit described by a body attached to an elastic thread, which is led through a fixed origin, which can be written

$$6r^{\frac{5}{2}}\cos\tfrac{5}{2}\theta = (4r^2 - ar + a^2)\sqrt{(2r+a)} \tag{12}$$

$$6r^{\frac{5}{2}}\sin\tfrac{5}{2}\theta = (r+a)\sqrt{(4r^3 - 8ar^2 + 2a^2r - a^3)} \tag{13}$$

So also for $\mu = 7$.

The Second Stage.

22. But in the dynamical applications, such as Poinsot's herpolhode and Jacobi's associated motion of the top, the integral of the Second Stage is required, corresponding to an even value of μ, and S can now be resolved into its factors

$$\mathrm{S} = 4(s - s_1)(s - s_2)(s - s_3) \tag{1}$$

while in most cases of the applications

$$s_1 > \sigma > s_2 > s > s_3 \tag{2}$$

so that

$$v = \omega_1 + f\omega_3, \quad u = f'\omega_1 + \omega_3 \tag{3}$$

where f, f' denotes real fractions.

To obtain this resolution of S we find it convenient to put (L.M.S., 27, p. 449),

$$x = -\frac{m^3\alpha}{(\alpha - m)^2}, \quad y = -\frac{(1-2m)\alpha}{\alpha - m} \tag{4}$$

and with $s = t^2$,

$$\mathrm{T}_1, \mathrm{T}_2 = \left(t \pm \frac{m\alpha}{\alpha - m}\right)\left\{2t^2 \mp \frac{mt}{\alpha - m} + \frac{m^2(1-2m)\alpha}{(\alpha-m)^2}\right\} \tag{5}$$

Denoting the roots of $\mathrm{S} = 0$, irrespective of order of magnitude, by a^2, b^2, c^2, we can put

$$a, b = \frac{m}{4(\alpha - m)}[1 \pm \sqrt{\{1 - 8(1-2m)\alpha\}}], \quad c = \frac{m\alpha}{\alpha - m} \tag{6}$$

$$a + b = \frac{m}{2(\alpha - m)}, \quad a - b = \frac{m\sqrt{\{1 - 8(1-2m)\alpha\}}}{2(\alpha - m)} \tag{7}$$

$$(a+c)(b+c) = \frac{m\alpha(\alpha - m + 1)}{(\alpha - m)^2}, \quad (a-c)(b-c) = \frac{m^3\alpha}{\alpha - m} \tag{8}$$

With $\frac{1}{2}\mu v$ congruent to a half-period ω, we can take

$$s(\tfrac{1}{2}\mu v) = s(\omega) = c^2 = \frac{m^2\alpha^2}{(\alpha - m)^2}, \quad \mathrm{S}(\tfrac{1}{2}\mu v) = 0 \tag{9}$$

and this leads to a relation between α and m, by means of which they are expressible theoretically in terms of a single variable.

Also, as shown in L.M.S., 27, p. 450,

$$s(\omega) - s(v) = \frac{m^2\alpha}{\alpha - m} \tag{10}$$

$$s(\omega) - s(2v) = \frac{m^2\alpha^2}{(\alpha - m)^2} \tag{11}$$

$$s(\omega) - s(3v) = \frac{(1-m)^2\alpha}{\alpha - m} \tag{12}$$

$$s(\omega) - s(4v) = \frac{m^2\{(1-2m)\alpha - m(1-m)\}^2}{(1-2m)^2(\alpha - m)^2} \tag{13}$$

$$s(\omega) - s(5v) = \frac{m^2\alpha}{\alpha - m}\left\{\frac{(1-2m)\alpha - m^2(1-m)}{(1-2m)\alpha - m(1-m)^2}\right\}^2$$
$$= \frac{m^2\alpha}{\alpha - m}\left(\frac{\mathrm{N}_5}{\mathrm{D}_5}\right)^2 \tag{14}$$

where

$$D_5 = (1 - 2m)\alpha - m(1-m)^2,$$

and N_5 is obtained from D_5 by a change of m into $1 - m$, and α into $-\alpha$.

$$s(\omega) - s(6v) = \frac{\alpha^2}{(\alpha - m)^2}\left(\frac{N_6}{D_6}\right)^2 \tag{15}$$

$$D_6 = 2(1-m)(1-2m)\alpha - m(1-m)^2,$$

$$N_6 = (1 - 2m + 2m^2)(1-2m)\alpha - m(1-m)(1 - 3m + 3m^2);$$

$$s(\omega) - s(7v) = \frac{m^2\alpha}{\alpha - m}\left(\frac{N_7}{D_7}\right)^2 \tag{16}$$

$$D_7 = (1-2m)^3\alpha^2 - m(1-m)(1-2m)(1-3m)\alpha - m^4(1-m)^2,$$

and N_7 is obtained by a change of m into $1 - m$ and α into $-\alpha$;

$$s(\omega) - s(8v) = \frac{m^2}{(1-2m)^2(\alpha - m)^2}\left(\frac{N_8}{D_8}\right)^2 \tag{17}$$

$$D_8 = \{(1-2m)\alpha - m(1-m)\}\{(1-2m)\alpha - m(1-m)(1-2m+2m^2)\},$$

$$N_8 = (1-2m)^3\alpha^3 - m(1-m)(1-2m)^2(1+2m-2m^2)\alpha^2 \\ + 4m^3(1-m)^3(1-2m)\alpha - m^4(1-m)^4;$$

$$s(\omega) - s(9v) = \frac{(1-m)^2\alpha}{\alpha - m}\left(\frac{N_9}{D_9}\right)^2 \tag{18}$$

$$D_9 = (1-2m)^3(3 - 6m + 4m^2)\alpha^3 - 2m(1-m)(1-2m)^2(3 - 7m + 5m^2)\alpha^2 \\ + m^2(1-m)^2(1-2m)(4 - 10m + 7m^2)\alpha - m^3(1-m)^6,$$

and a change of α into $-\alpha$ and m into $1 - m$ gives N_9;

$$s(\omega) - s(10v) = \frac{m^2\alpha^2}{(\alpha - m)^2}\left(\frac{N_{10}}{D_{10}}\right)^2 \tag{19}$$

$$D_{10} = \{(1-2m)\alpha - m(1-m)^2\}\{(1-2m)\alpha - m^2(1-m)\} \\ \{(1-2m)^4\alpha^2 - m(1-m)(1-2m)(1 - 6m + 6m^2)\alpha - m^3(1-m^3)\},$$

$$N_{10} = (1-2m)^6\alpha^4 - 2m(1-m)(1-2m)^3(2 - 7m + 7m^2)\alpha^3 \\ + 2m^2(1-m)^2(1-2m)^2(3 - 11m + 13m^2 - 4m^3 + 2m^4)\alpha^2 \\ - m^3(1-m)^3(1-2m)(4 - 17m + 27m^2 - 20m^3 + 10m^4)\alpha \\ + m^4(1-m)^4(1 - 5m + 10m^2 - 10m^3 + 5m^4).$$

A simplification is effected by putting

$$(1-2m)\alpha = m(1-m)\beta, \quad \text{and} \quad 1 - 2m = p \tag{20}$$

and now the change of m into $1 - m$ and α into $-\alpha$ is equivalent to a change of sign of p, leaving β unchanged.

Thus we find, with $2\beta - 1 = \epsilon$,

$$s(\omega) - s(11v) = \frac{m^2\alpha}{\alpha - m}\left(\frac{N_{11}}{D_{11}}\right)^2 \tag{21}$$

$$D_{11} = -p^5 + \epsilon(\epsilon - 2)(\epsilon^2 - \epsilon - 1)p^4 + \epsilon(\epsilon^2 + 1)p^3 + \epsilon^2(\epsilon^2 - 3)p^2 - \epsilon^3(\epsilon^2 - \epsilon + 1)p + \epsilon^3,$$

$$N_{11} = \quad p^5 + \epsilon(\epsilon - 2)(\epsilon^2 - \epsilon - 1)p^4 - \epsilon(\epsilon^2 + 1)p^3 + \epsilon^2(\epsilon^2 - 3)p^2 + \epsilon^3(\epsilon^2 - \epsilon + 1)p + \epsilon^3,$$

$$\begin{aligned} N_{11} - D_{11} &= 2p\{p^4 - \epsilon(\epsilon^2 + 1)p^2 + \epsilon^3(\epsilon^2 - \epsilon + 1)\}, \\ &= 2p(p^2 - \epsilon^2)(p^2 - \epsilon^3 + \epsilon^2 - \epsilon), \end{aligned}$$

$$\begin{aligned} N_{11} + D_{11} &= 2\epsilon\{(\epsilon - 2)(\epsilon^2 - \epsilon - 1)p^4 + \epsilon(\epsilon^2 - 3)p^2 + \epsilon^2\}, \\ &= 2\epsilon\{(\epsilon - 2)p^2 + \epsilon\}\{(\epsilon^2 - \epsilon - 1)p^2 + \epsilon\}. \end{aligned}$$

Thus, if $\mu = 22$, $s(11v) = s(\omega)$, and $N_{11} = 0$, which replaces in a much simpler form the relation $\gamma_{22} = 0$.

So also $D_{11} = 0$ makes $s(11v) = \infty$, $v = \frac{2\omega}{11}$, and so enables us to connect up the results for $\mu = 11$; and other values of μ can be treated in the same way.

23. There are three cases to consider, relative to the half-period ω, to which $\frac{1}{2}\mu v$ is congruent :—

I. $$\omega = \omega_1, \quad s(\tfrac{1}{2}\mu v) = s(\omega_1) = s_1 = c^2, \quad s_2 = a^2, \quad s_3 = b^2 \tag{1}$$

and introducing WEBER's function $f\omega$, or KIEPERT's equivalent function $L(2)$,

$$L(2)^{24} = (f_1\omega)^{24} = 16\frac{\kappa^4}{\kappa'^2} = \frac{16(s_2 - s_3)^2}{(s_1 - s_3)(s_1 - s_2)} = \frac{1 - 8(1 - 2m)\alpha}{\alpha^2(\alpha - m)(\alpha - m + 1)} \tag{2}$$

and to the complementary modulus κ',

$$\frac{1}{\text{cn}^2 f\text{K}'}, \quad \frac{1}{\text{dn}^2 f\text{K}'} = \frac{1 + 4\alpha \pm \sqrt{\{1 - 8(1 - 2m)\alpha}}{2} \tag{3}$$

$$\text{cn}2f\text{K}' = \frac{b}{c}, \quad \text{dn}2f\text{K}' = \frac{a}{c}, \quad \text{sn}(1 - 2f)\text{K}' = \frac{b}{a} \tag{4}$$

II. $$\omega = \omega_2, \quad s(\tfrac{1}{2}\mu v) = s(\omega_2) = s_2 = c^2, \quad s_1 = a^2, \quad s_3 = b^2 \tag{5}$$

$$(f\omega)^{24} = \frac{16}{\kappa^2\kappa'^2} = \frac{16(s_1 - s_3)^2}{(s_2 - s_3)(s_1 - s_2)} = \frac{-1 + 8(1 - 2m)\alpha}{\alpha^2(\alpha - m)(\alpha - m + 1)} = -L(2)^{24} \tag{6}$$

$$\frac{1}{\text{sn}^2 f\text{K}'}, \quad \text{dn}^2(1 - f)\text{K}' = \frac{1 + 4\alpha \pm \sqrt{\{1 - 8(1 - 2m)\alpha\}}}{2} \tag{7}$$

$$\text{cn}2f\text{K}' = \frac{a}{b}, \quad \text{dn}2f\text{K}' = \frac{c}{a}, \quad \text{sn}(1 - 2f)\text{K}' = \frac{b}{c} \tag{8}$$

III. $$\omega = \omega_3, \quad s\left(\tfrac{1}{2}\mu v\right) = s(\omega_3) = s_3 = c^2, \quad s_1 = a^2, \quad s_2 = b^2 \qquad (9),$$

$$(f_2\omega)^{24} = 16\frac{\kappa'^4}{\kappa^2} = 16\frac{(s_1 - s_2)^2}{(s_1 - s_3)(s_2 - s_3)} = \frac{1 - 8(1 - 2m)\alpha}{\alpha^2(\alpha - m)(\alpha - m + 1)} = \mathrm{L}(2)^{24} \qquad (10),$$

$$\frac{1}{\kappa'^2 \mathrm{sn}^2 f\mathrm{K}'}, \quad -\frac{\kappa^2}{\kappa'^2 \mathrm{cn}^2 f\mathrm{K}'} = \frac{1 - 4\alpha \pm \sqrt{\{1 - 8(1 - 2m)\alpha\}}}{2}. \qquad (11).$$

$$\mathrm{cn}2f\mathrm{K}' = \frac{c}{a}, \quad \mathrm{dn}2f\mathrm{K}' = \frac{b}{a}, \quad \mathrm{sn}(1 - 2f)\mathrm{K}' = \frac{c}{b} \qquad (12).$$

A fourth case, where $1 - 8(1 - 2m)\alpha$ is negative and a, b imaginary, need not be discussed here.

The Weierstrassian Sigma Functions of the Division-values (Theilwerthe) can now be expressed in terms of the Jacobian Theta and Eta Functions by the relations on p. 52 of SCHWARZ, 'Formeln der elliptischen Functionen,' so that

$$\mu = 4n + 2, \quad v = \mathrm{K} + \frac{\mathrm{K}'i}{2n + 1} \qquad (13),$$

$$\frac{\Theta\dfrac{\mathrm{K}'}{2n+1}}{\Theta\mathrm{K}'} = \sqrt[4]{\{(s_1 - s_2)(s_2 - s_3)\}}\, x^{-\frac{1}{3}}\lambda^{-\frac{1}{4n+2}} \qquad (14),$$

$$\frac{\mathrm{H}\dfrac{2\mathrm{K}'}{2n+1}}{\Theta\mathrm{K}'} = \sqrt[4]{\{(s_1 - s_2)(s_2 - s_3)\}}\, x^{-\frac{1}{3}}\lambda^{-\frac{4}{4n+2}} \qquad (15),$$

$$\frac{\Theta\dfrac{2\mathrm{K}'}{2n+1}}{\Theta 0} = \sqrt{(s_1 - s_3)}\, x^{-\frac{1}{3}}\lambda^{-\frac{4}{4n+2}}\,\mathrm{ns}\frac{2\mathrm{K}'}{2n+1}$$

$$= x^{-\frac{1}{3}}\lambda^{-\frac{4}{4n+2}}\frac{ma}{\alpha - m} \qquad (16).$$

$$\mu = 4n, \quad v = \mathrm{K} + \frac{\mathrm{K}'i}{2n} \qquad (17),$$

$$\frac{\Theta\dfrac{\mathrm{K}'}{2n}}{\mathrm{H}\mathrm{K}'} = \sqrt[4]{\{(s_1 - s_3)(s_2 - s_3)\}}\, x^{-\frac{1}{3}}\lambda^{-\frac{1}{4n}} \qquad (18),$$

$$\frac{\mathrm{H}\dfrac{\mathrm{K}'}{n}}{\mathrm{H}\mathrm{K}'} = \sqrt[4]{\{(s_1 - s_3)(s_2 - s_3)\}}\, x^{-\frac{1}{3}}\lambda^{-\frac{1}{n}} \qquad (19),$$

$$\frac{\Theta\dfrac{\mathrm{K}'}{n}}{\Theta 0} = x^{-\frac{1}{3}}\lambda^{-\frac{1}{n}}\frac{m\alpha}{\alpha - m} \qquad (20).$$

24. When μ is of the form $4n + 2$, so that, as required in dynamical problems,

$$v = \omega_1 + \frac{r\omega_3}{2n+1}, \quad \text{or} \quad \mathrm{K} + \frac{r\mathrm{K}'i}{2n+1} \tag{1},$$

then $\frac{1}{2}\mu v = (2n + 1)v$ is congruent to ω_1 or ω_2, according as r is even or odd; and the integral $\mathrm{I}(v)$ is expressible in the form

$$\mathrm{I}(v) = \int_{b^2}^{s} \frac{-\mathrm{P}(v)(\sigma - s) + \frac{1}{2}x}{\sigma - s}\frac{ds}{\sqrt{\mathrm{S}}}$$

$$= \frac{1}{2n+1}\cos^{-1}\frac{s^n + \mathrm{P}_1 s^{n-1} + \ldots + \mathrm{P}_n}{(\sigma - s)^{n+\frac{1}{2}}}\sqrt{(c^2 - s)}$$

$$= \frac{1}{2n+1}\sin^{-1}\frac{\mathrm{Q}_1 s^{n-1} + \ldots + \mathrm{Q}_n}{(\sigma - s)^{n+\frac{1}{2}}}\sqrt{(a^2 - s \,.\, s - b^2)} \tag{2},$$

and when the quantities α, m and therefore a, b, c, σ, x, and $\mathrm{P}(v)$ have been determined as algebraical functions of a parameter, the calculation of $\mathrm{P}_1, \ldots \mathrm{P}_n$, $\mathrm{Q}_1, \ldots, \mathrm{Q}_n$, is effected readily by a differentiation and verification.

It will be found that

$$\mathrm{Q}_1 = (2n + 1)\,\mathrm{P}(v) \tag{3},$$

also

$$\frac{\mathrm{P}(\omega_1 + f\omega_3)}{\sqrt{(s_1 - s_3)}} = \mathrm{zn}\,(f\mathrm{K}', \kappa'), \quad \frac{\mathrm{P}(f\omega_3)}{\sqrt{(s_1 - s_3)}} = \mathrm{zs}f\mathrm{K}' \tag{4},$$

$\mathrm{zn}u$ denoting Jacobi's Zeta function $\mathrm{Z}u$ or $\dfrac{d}{du}\log\Theta u$, while

$$\mathrm{zs}u = \mathrm{zn}u + \frac{d}{du}\log \mathrm{sn}u = \frac{d}{du}\log \mathrm{H}u. \tag{5}$$

in Glaisher's notation; also

$$\mathrm{P}(2n + 1)v = 0, \quad \mathrm{P}(v) + \mathrm{P}(2nv) = \frac{-\frac{1}{2}x}{\sigma - s(\omega)} = -\frac{1}{2}\frac{m}{\alpha - m} \tag{6}.$$

Here already the degree of the results obtained by Abel's method have been halved in (2); and the degree can be halved again by the substitution

$$y^2 = \frac{\sigma - s}{c^2 - s} \tag{7},$$

and now

$$\mathrm{I}(v) = \frac{1}{2n+1}\cos^{-1}\frac{s^n + \mathrm{P}_1 s^{n-1} + \ldots + \mathrm{P}_n}{(\sigma - s)^n\, y}$$

$$= \frac{1}{2n+1}\cos^{-1}\frac{\mathrm{B}_0 y^{2n} + \mathrm{B}_1 y^{2n-2} + \ldots + \mathrm{B}_n}{y^{2n+1}}$$

$$= \frac{2}{2n+1}\cos^{-1}\frac{y^{n-1} + \mathrm{A}_1 y^{n-2} + \ldots + \mathrm{A}_{n-1}}{y^{n+\frac{1}{2}}}\sqrt{\{\tfrac{1}{2}(y+1)(y^2 + \mathrm{C}_1 y + \mathrm{C}_2)\}}$$

$$= \frac{2}{2n+1}\sin^{-1}\frac{y^{n-1} - \mathrm{A}_1 y^{n-2} + \ldots + (-1)^{n-1}\mathrm{A}_{n-1}}{y^{n+\frac{1}{2}}}\sqrt{\{\tfrac{1}{2}(y-1)(y^2 - \mathrm{C}_1 y + \mathrm{C}_2)\}} \tag{8}.$$

Denoting by a'^2, b'^2 the values of y^2 corresponding to a^2, b^2 of s,

$$a'^2 = \frac{\sigma - a^2}{c^2 - a^2}, \quad b'^2 = \frac{\sigma - b^2}{c^2 - b^2} \tag{9}$$

reducing to

$$a'^2, b'^2 = \frac{2}{1 + 4\alpha \pm \sqrt{\{1 - 8(1 - 2m)\alpha\}}} \tag{10}$$

so that

$$a' = \text{cn} f \text{K}', \quad b' = \text{dn} f \text{K}', \qquad \text{in Region I.} \tag{11}$$

$$a' = \text{sn} f \text{K}', \quad b' = \frac{1}{\text{dn}(1 - f)\text{K}'}, \text{ in Region II.} \tag{12}$$

Changing the variable to y in the integral

$$\text{I}(v') = \int_{1, \text{ or } b'}^{y} \frac{\text{P}(v')y^2 - \text{Q}(v')}{y^2} \frac{dy^2}{\sqrt{\text{Y}}} \tag{13}$$

where

$$\text{Y} = 4(y^2 - 1)(y^2 - a'^2)(y^2 - b'^2) \tag{14}$$

$$\text{P}(v') = \frac{\text{P}(\omega - v)}{\text{M}'}, \quad \text{M}'^2 = \frac{m^2\alpha(\alpha - m + 1)}{(\alpha - m)^2} \tag{15}$$

$$\text{Q}(v') = \frac{\frac{1}{2}is'(\omega - v)}{\text{M}'^3} = a'b' = \text{C}_2 \tag{16}$$

$$\text{C}_2{}^2 = a'^2b'^2 = \frac{1}{4\alpha(\alpha - m + 1)} \tag{17}$$

$$\text{C}_1{}^2 - 2\text{C}_2 = a'^2 + b'^2 = \frac{1 + 4\alpha}{4\alpha(\alpha - m + 1)} \tag{18}$$

$$\kappa^2 = \frac{1 - a'^2}{b'^2 - a'^2}, \quad \kappa'^2 = \frac{b'^2 - 1}{b'^2 - a'^2} \tag{19}$$

$$\kappa^2 - \kappa'^2 = \frac{-1 + 4(1 - 2m)\alpha + 8\alpha^2}{\sqrt{1 - 8(1 - 2m)\alpha}} \tag{20}$$

25. To connect up ABEL's results for even values of μ we take $n = 1$ in (9), § 5, so that

$$a = \frac{2x - y - y^2}{y}, \quad b = \frac{-x(y - x - y^2)}{y^2}, \quad p = 4xy \tag{1}$$

$$\begin{aligned} \text{Z} &= -(z^2 - az + b)^2 + pz \\ &= -(z^2 - az + b + 2s)^2 + 4(\text{A}z + \text{B})^2 \end{aligned} \tag{2}$$

where

$$\text{A}^2 = s, \quad 2\text{AB} = -as + xy, \quad \text{B}^2 = s(s + b) \tag{3}$$

and therefore the resolving cubic becomes

$$4s^2(s + b) - (-as + xy)^2 = 0 \tag{4}$$

or

$$S = 4s(s + x)^2 - [(1 + y)s + xy]^2 = 0 \tag{5}$$

Then, with

$$x = -\frac{m^3\alpha}{(\alpha - m)^2}, \quad y = -\frac{(1 - 2m)\alpha}{\alpha - m}, \quad s = \frac{m^2\alpha^2}{(\alpha - m)^2} \tag{6}$$

$$Az + B = \frac{m\alpha}{\alpha - m}\left[z + \frac{m(1 - 2m)\alpha - m^2(1 - m)}{(1 - 2m)(\alpha - m)}\right] \tag{7}$$

and Z breaks up into two quadratic factors, Z_1 and Z_2,

$$\begin{aligned} Z_1 &= z^2 - \frac{m(1 - 2m + 2m^2)}{(1 - 2m)(\alpha - m)}z + \frac{m^7(1 - m)^2}{(1 - 2m)^2(\alpha - m)^2} \\ &= \left[z - \frac{m^3}{(1 - 2m)(\alpha - m)}\right]\left[z - \frac{m(1 - m)^2}{(1 - 2m)(\alpha - m)}\right] \end{aligned} \tag{8}$$

$$Z_2 = z^2 + \frac{m\{4(1 - 2m)\alpha - 1 + 2m - 2m^2\}}{(1 - 2m)(\alpha - m)}z + \frac{m^2[2(1 - 2m)\alpha - m(1 - m)]^2}{(1 - 2m)^2(\alpha - m)^2} \tag{9}$$

$$Z_1 = \left[z - \frac{m^2(1 - m)}{(1 - 2m)(\alpha - m)}\right]^2 - \frac{m(1 - 2m)}{\alpha - m}z \tag{10}$$

$$Z_2 = \left[z + \frac{2m(1 - 2m)\alpha - m^2(1 - m)}{(1 - 2m)(\alpha - m)}\right]^2 - \frac{m(1 - 2m)}{\alpha - m}z \tag{11}$$

and by means of a homogeneity factor

$$M = \frac{(1 - 2m)(\alpha - m)}{m} \tag{12}$$

replacing z by $\frac{z}{M}$, we may replace Z_1 and Z_2 by

$$\begin{aligned} Z_1 &= \{z - m(1 - m)\}^2 - (1 - 2m)^2 z \\ &= (z - m^2)\{z - (1 - m)^2\} \end{aligned} \tag{13}$$

$$Z_2 = \{z + 2(1 - 2m)\alpha - m(1 - m)\}^2 - (1 - 2m)^2 z \tag{14}$$

26. The resolution of T in the spherical catenary for $\mu = 4n + 2$ is not the same as for $\mu = 2n + 1$; we must put

$$T_1, T_2 = \mp(1 + 2\lambda t + t^2)^2 + 2t(\mu + \nu t^2) \tag{1}$$

and then in § 20

$$\mu^2 = \lambda - 1 + \left(\frac{h - 1}{A}\right)^2, \quad 2\mu\nu = \lambda^2 - 1 + 2\frac{h^2 - 1}{A^2}, \quad \nu^2 = \lambda - 1 + \left(\frac{h + 1}{A}\right)^2 \tag{2}$$

so that

$$\left(\lambda^2 - 1 + 2\frac{h^2 - 1}{A^2}\right)^2 - 4\left\{\lambda - 1 + \left(\frac{h - 1}{A}\right)^2\right\}\left\{\lambda - 1 + \left(\frac{h + 1}{A}\right)^2\right\} = 0 \tag{3}$$

a quartic for λ, having a root $\lambda = 1$, which was used for $\mu = 2n + 1$; the remaining cubic, putting $\lambda = 2B - 1$, becomes

$$B^3 - B^2 + \left(\frac{h^2 - 1}{A^2} - 1\right)B - \frac{h^2 + 1}{A^2} + 1 = 0 \qquad (4)$$

or as in § 20 with

$$\frac{h^2 - 1}{A^2} = 1 + \frac{4x}{(y+1)^2}, \quad \frac{h^2 + 1}{A^2} = 1 + \frac{4xy}{(y+1)^3} \qquad (5),$$

$$B^3 - B^2 + \frac{4x}{(y+1)^2}B - \frac{4xy}{(y+1)^3} = 0 \qquad (6);$$

so that $B = \frac{2c}{y+1}$ reduces this to

$$2c^3 - (y+1)c^2 + 2xc - x = 0 \qquad (7),$$

as in (3), § 8; and with

$$x = -\frac{m^3\alpha}{\alpha - m}, \quad y = -\frac{(1-2m)\alpha}{\alpha - m}, \quad y + 1 = \frac{(2\alpha - 1)m}{\alpha - m} \qquad (8),$$

$$c = \frac{m\alpha}{\alpha - m}, \quad B = \frac{2\alpha}{2\alpha - 1}, \quad \lambda = \frac{2\alpha + 1}{2\alpha - 1} \qquad (9).$$

We find that CLEBSCH'S k (CRELLE, 57, p. 105; L.M.S., 27, pp. 146, 185) is given by

$$k^2 = 4\alpha - 1 \qquad (10),$$

so that

$$\lambda = \frac{k^2 + 3}{k^2 - 1} \qquad (11),$$

and

$$A^2 = \frac{(k^2 - 1)^2(k^2 - h^2)}{4k^2} \qquad (12),$$

so that

$$\mu^2 = 4\frac{(k^2 - h)^2}{(k^2 - 1)^2(k^2 - h^2)}, \quad \nu^2 = 4\frac{(k^2 + h)^2}{(k^2 - 1)^2(k^2 - h^2)} \qquad (13);$$

and in quadratic factors

$$(k^2-1)T_1 = -\left\{k+1-2\sqrt{\frac{k+h}{k-h}}\,t+(k-1)t^2\right\}\left\{k-1-2\sqrt{\frac{k-h}{k+h}}\,t+(k+1)t^2\right\} \qquad (14),$$

$$(k^2-1)T_2 = \left\{k+1+2\sqrt{\frac{k+h}{k-h}}\,t+(k-1)t^2\right\}\left\{k-1+2\sqrt{\frac{k-h}{k+h}}\,t+(k+1)t^2\right\} \qquad (15).$$

Thus, as shown in the 'Bulletin de la Société Mathématique de France,' 29, the algebraical spherical catenary for $\mu = 10$, drawn stereoscopically by the late Mr. T. I. DEWAR, can be represented, by taking $k = \sqrt{\frac{5}{3}}$, $h = -\frac{1}{2}\sqrt{\frac{17}{3}}$, in the symmetrical form

$$\begin{aligned} \mathrm{N}\,(\tan\tfrac{1}{2}\theta e^{\psi i})^{\frac{1}{3}} &= (\mathrm{B} + \mathrm{B}_1 t + \mathrm{B}_2 t^2 + \mathrm{B}_3 t^3)\sqrt{\mathrm{T}_1} \\ &\quad + i\,(\mathrm{B} - \mathrm{B}_1 t + \mathrm{B}_2 t^2 - \mathrm{B}_3 t^3)\sqrt{\mathrm{T}_2} \end{aligned} \qquad (16),$$

$$\mathrm{B},\ \mathrm{B}_3 = \pm\sqrt{51} + 7, \quad \mathrm{B}_1,\ \mathrm{B}_2 = \pm 3\sqrt{51} + 19, \quad \mathrm{N} = 192\sqrt{3} \qquad (17).$$

27. These expressions for $\mathrm{I}(v)$ are applicable immediately to the construction of a series of quasi-algebraical Poinsot herpolhodes, in continuation of the one invented by HALPHEN ('F.E.,' 2, p. 279).

Making a digression on the motion of a rigid body about a fixed point under no forces, as illustrated by the motion of a body about its centre of gravity when tossed in the air, POINSOT'S polhode and herpolhode are obtained by rolling the momental ellipsoid on the invariable plane.

The equation of the momental ellipsoid can be written

$$\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 = \mathrm{D}\delta^2 \qquad (1),$$

where δ denotes the distance of the invariable plane from the centre of the momental ellipsoid; and then the polhode will be the intersection with the coaxial quadric

$$\mathrm{A}^2x^2 + \mathrm{B}^2y^2 + \mathrm{C}^2z^2 = \mathrm{D}^2\delta^2 \qquad (2),$$

and the direction cosines of the invariable line will be

$$\frac{\mathrm{A}x}{\mathrm{D}\delta},\quad \frac{\mathrm{B}y}{\mathrm{D}\delta},\quad \frac{\mathrm{C}z}{\mathrm{D}\delta} \qquad (3).$$

Denoting by ρ, ϖ the polar co-ordinates for the herpolhode in the invariable plane,

$$x^2 + y^2 + z^2 = \rho^2 + \delta^2 \qquad (4),$$

and by solution of these equations (1), (2), (4),

$$\begin{aligned} \left(1 - \frac{\mathrm{A}}{\mathrm{B}}\right)\left(1 - \frac{\mathrm{A}}{\mathrm{C}}\right)x^2 &= \rho^2 + \left(1 - \frac{\mathrm{D}}{\mathrm{B}}\right)\left(1 - \frac{\mathrm{D}}{\mathrm{C}}\right)\delta^2 \\ &= \rho^2 - \rho_a{}^2 \text{ suppose, \&c.} \end{aligned} \qquad (5),$$

and then

$$\left\{\frac{(\mathrm{B} - \mathrm{C})(\mathrm{C} - \mathrm{A})(\mathrm{A} - \mathrm{B})}{\mathrm{ABC}}\,xyz\right\}^2 = \tfrac{1}{4}\mathrm{R} \qquad (6),$$

where

$$\mathrm{R} = 4\,(\rho_a{}^2 - \rho^2)(\rho_b{}^2 - \rho^2)(\rho_c{}^2 - \rho^2) \qquad (7).$$

Denoting the component angular velocities about the principal axes by p, q, r, and about the invariable line by h,

$$\frac{p}{x} = \frac{q}{y} = \frac{r}{z} = \frac{h}{\delta} = \frac{n}{k} \qquad (8),$$

and DARBOUX's equations (DESPEYROUS, 'Cours de mécanique,' II., note 17).

$$\frac{p^2}{a} + \frac{q^2}{b} + \frac{r^2}{c} = h, \qquad \frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2} = 1 \tag{9}$$

are identified with our notation by putting

$$\mathrm{A}a = \mathrm{B}b = \mathrm{C}c = \mathrm{D}h \tag{10}$$

and

$$\mathrm{A}p^2 + \mathrm{B}q^2 + \mathrm{C}r^2 = \mathrm{D}h^2 \tag{11}$$

$$\mathrm{A}^2p^2 + \mathrm{B}^2q^2 + \mathrm{C}^2r^2 = \mathrm{D}^2h^2 \tag{12}$$

are two integrals of EULER's equations

$$\mathrm{A}\frac{dp}{dt} = (\mathrm{B} - \mathrm{C})\,qr, \ldots \tag{13}$$

or, as they may be written,

$$\mathrm{A}\frac{dx}{dt} = n\,(\mathrm{B} - \mathrm{C})\,\frac{yz}{k}, \ldots \tag{14}$$

Then

$$\begin{aligned}
\frac{d\rho^2}{dt} &= 2\left(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right) \\
&= 2n\left(\frac{\mathrm{B}-\mathrm{C}}{\mathrm{A}} + \frac{\mathrm{C}-\mathrm{A}}{\mathrm{B}} + \frac{\mathrm{A}-\mathrm{B}}{\mathrm{C}}\right)\frac{xyz}{k} \\
&= -\,2n\,\frac{(\mathrm{B}-\mathrm{C})\,(\mathrm{C}-\mathrm{A})\,(\mathrm{A}-\mathrm{B})}{\mathrm{ABC}}\,\frac{xyz}{k} = \frac{n}{k}\sqrt{\mathrm{R}}
\end{aligned} \tag{15}$$

$$nt = \int_{\rho_2}^{\rho} \frac{k\,d\rho^2}{\sqrt{\mathrm{R}}} \tag{16}$$

supposing

$$\rho_3^{\,2} > \rho^2 > \rho_2^{\,2} > 0 > \rho_1^{\,2} \tag{17}$$

and inverting the integral,

$$\rho^2 = \rho_2^{\,2}\mathrm{sn}^2\,(\mathrm{K} - mt) + \rho_3^{\,2}\mathrm{cn}^2\,(\mathrm{K} - mt), \qquad \frac{m}{n} = \sqrt{\frac{\rho_3^{\,2} - \rho_1^{\,2}}{k^2}} \tag{18}$$

Again, projecting on the invariable plane the areas swept out on the co-ordinate planes

$$\rho^2\frac{d\varpi}{dt} = \frac{\mathrm{A}x}{\mathrm{D}\delta}\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) + \ldots + \ldots \tag{19}$$

and

$$\begin{aligned}
y\frac{dz}{dt} - z\frac{dy}{dt} &= \frac{n}{k}\left(\frac{\mathrm{A}-\mathrm{B}}{\mathrm{C}}\,xy^2 + \frac{\mathrm{A}-\mathrm{C}}{\mathrm{B}}\,xz^2\right) \\
&= \frac{n\delta^2}{k}\,\frac{\mathrm{D}\,(\mathrm{A}-\mathrm{D})}{\mathrm{BC}}\,x
\end{aligned} \tag{20}$$

so that

$$\rho^2 \frac{d\varpi}{dt} = \frac{n\delta}{k}\left(\frac{A-D}{BC} Ax^2 + \frac{B-D}{CA} By^2 + \frac{C-D}{AB} Cz^2\right)$$

$$= \frac{n\,\delta}{k} \frac{A^3x^2 + B^3y^2 + C^3z^2 - D^3\,\delta^2}{ABC} \tag{21}$$

and this reduces to

$$\rho^2 \frac{d\varpi}{dt} = \frac{n\,\delta}{k}\left\{\rho^2 + \frac{(A-D)(B-D)(C-D)}{ABC}\,\delta^2\right\} \tag{22}$$

so that

$$\varpi = \frac{\delta}{k} nt + \frac{(A-D)(B-D)(C-D)}{ABC}\,\delta^3 \int_{\rho_2}^{\rho} \frac{d\rho^2}{\rho^2 \sqrt{R}} \tag{23}$$

Putting

$$\frac{\rho^2}{k^2} = \frac{\sigma - s}{M^2}, \quad \frac{\rho_a{}^2 - \rho^2}{k^2} = \frac{s - s_a}{M^2}, \quad \frac{R}{k^6} = \frac{S}{M^6} \tag{24}$$

and $\rho = 0$, with $s = \sigma$,

$$R_0 = 4\rho_a{}^2\rho_b{}^2\rho_c{}^2 = -4\left\{\frac{(A-D)(B-D)(C-D)}{ABC}\right\}^2 \delta^6 = \frac{k^6\Sigma}{M^6} \tag{25}$$

so that

$$\frac{(A-D)(B-D)(C-D)}{ABC}\,\delta^3 = -\frac{\frac{1}{2}k^3\sqrt{-\Sigma}}{M^3} \tag{26}$$

and from (6) § 1

$$\varpi = \frac{\delta}{k} nt - \int_s^{s_2} \frac{\frac{1}{2}\sqrt{-\Sigma}}{\sigma - s} \frac{ds}{\sqrt{S}}$$

$$= \left(\frac{\delta}{k} - \frac{Pv}{M}\right) nt - I(v) \tag{27}$$

or

$$pt - \varpi = I(v) \tag{28}$$

where

$$\frac{p}{n} = \frac{\delta}{k} - \frac{Pv}{M} \tag{29}$$

also

$$nt = \int_s^{s_2} \frac{M\,ds}{\sqrt{S}} \tag{30}$$

Then

$$e^{2iI(v)} = \frac{\theta(u-v)}{\theta(u+v)} \tag{31}$$

and

$$M^2 \frac{\rho^2}{k^2} = \sigma - s = \wp v - \wp u$$

$$= \frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u \sigma^2 v} = \frac{\theta^2 0 \theta(u+v)\,\theta(u-v)}{\lambda^2\theta^2 u \theta^2 v} \tag{32}$$

where λ is a homogeneity factor, $\lambda = m/n$; so that as a quasi-Lamé function

$$M \frac{\rho}{k} e^{(pt-\varpi)i} = \frac{\theta 0 \theta(u-v)}{\lambda\theta u \theta v}$$

$$= \{(s^n + P_1 s^{n-1} + \ldots + P_n)\sqrt{(c^2 - s)}$$

$$+ i(Q_1 s^{n-1} + \ldots + Q_n)\sqrt{(a^2 - s \,.\, s - b^2)}\}^{\frac{1}{2n+1}} \tag{33}$$

in the pseudo-elliptic case; and cancelling the secular term pt by making $p = 0$ gives a purely algebraical herpolhode.

With the degree halved by the use of the variable y, as in (8) § 24,

$$\mathrm{M}\frac{\rho}{k}e^{(pt-\varpi)i} = \frac{[(y^{n-1}+\mathrm{A}_1y^{n-2}+\ldots)\sqrt{\mathrm{Y}_1}+i(y^{n-1}-\mathrm{A}_1y^{n-2}+\ldots)\sqrt{\mathrm{Y}_2}]^{\frac{2}{2n+1}}}{\sqrt{(y^2-1)}} \qquad (34),$$

$$\mathrm{M}^2\frac{\rho^2}{k^2} = \frac{y^2}{y^2-1} \qquad (35).$$

In the associated motion of a top, the projection on a horizontal plane of a point on the axis describes the hodograph of a herpolhode traced out by the axis of resultant angular momentum of the top; and thus EULER's angles θ and ψ for the top are connected by a relation of the form

$$\tfrac{1}{2}\mathrm{M}^2\sin\theta e^{\psi i} = -i\frac{d}{dt}(\rho e^{\varpi i}) \qquad (36)$$

('Science,' Dec. 20, 1901; 'Annals of Mathematics,' vol. 5, Series II., 1903).

We may dispense with POINSOT's rolling surface and consider the polhode as a material line, or as the edge of a material cone, as in Dr. FR. SCHILLING's model, constructed by MARTIN SCHILLING, Halle a. S., and this is rolled on a fixed plane.

According to M. DE LA GOURNERIE, the polhode is also a line of curvature, the intersection of two confocal surfaces, an ellipsoid and hyperboloid of two sheets; and θ will now be the angle between the generating lines of the hyperboloid of one sheet, one generator being perpendicular to the fixed plane, so that the other generator moves parallel to the axis of the top in the associated motion (DARBOUX in DESPEYROUS, 'Mécanique,' note 18).

28. To utilise for $\mu = 4n+2$ the preceding results for odd order $2n+1$ of μ, where

$$t = \frac{m\alpha}{\alpha - m}y \qquad (1),$$

$$\mathrm{T}_1, \mathrm{T}_2 = \frac{m^3\alpha^3}{(\alpha-m)^3}(y\pm 1)\left[2y^2 \mp \frac{y}{\alpha} + \frac{(1-2m)}{\alpha}\right] \qquad (2),$$

put

$$s(\omega) - s(4n+2)v = \infty, \quad \mathrm{D}_{2n+1} = 0 \qquad (3),$$

and thence express $\alpha, m, \ldots$ in terms of a new parameter, and thereby express the integral $\mathrm{I}(4v)$ of $\mu = 4n+2$.

Returning then to the v of $\mu = 2n+1$, which, when normalised to LEGENDRE's form, can be written

$$v = f\mathrm{K}'i, \quad f = \frac{2m}{2n+1} \qquad (4),$$

and replacing $\gamma_\mu = 0$ by its equivalent $\mathrm{D}_{2n+1} = 0$, the series of functions

$$a_1 = 1 + \frac{1}{\alpha - m} \tag{5}$$

$$a_2 = 1 - \frac{m-1}{\alpha} \tag{6}$$

$$a_3 = \left(1 + \frac{1}{\alpha - m}\right)\frac{m^2}{(1-m)^2} \tag{7}$$

$$a_4 = \left(1 - \frac{m-1}{\alpha}\right)\left\{\frac{(1-2m)\,\alpha}{(1-2m)\,\alpha - m\,(1-m)}\right\}^2 \tag{8}$$

. . . .

$$a_{2n} = \left(1 - \frac{m-1}{\alpha}\right)\left(\frac{D_{2n}}{N_{2n}}\right)^2 \tag{9}$$

$$a_{2n+1} = \left(1 + \frac{1}{\alpha - m}\right)\left(\frac{D_{2n+1}}{N_{2n+1}}\right)^2 \tag{10}$$

is such that

$$a_r = \kappa^2 \mathrm{tn}^4 rf\mathrm{K}' = \left[\frac{\mathrm{H}\,(rf\mathrm{K}')}{\mathrm{H}\,(1-rf)\,\mathrm{K}'}\right]^4, \text{ or } -\frac{\kappa'^2}{\kappa^2}\mathrm{cn}^4(1-rf)\mathrm{K}' = -\left[\frac{\mathrm{H}\,(rf\mathrm{K}')}{\Theta\,(1-rf)\,\mathrm{K}'}\right]^4 \tag{11}$$

so that JACOBI's division-values of the Second Stage can be calculated.

The Weierstrassian Sigma Functions of the First Stage in § 4 can now be converted into Jacobian Theta Functions of the Second Stage by the relations on p. 52 of 'Formeln und Lehrsätze'; and forming the series of functions

$$c_1 = x^{-\frac{1}{3}}\lambda^{-\frac{1}{2n+1}}\, m\sqrt{\frac{\alpha}{\alpha - m}} \tag{12}$$

$$c_2 = x^{-\frac{1}{3}}\lambda^{-\frac{4}{2n+1}}\,\frac{m\alpha}{\alpha - m} \tag{13}$$

$$c_3 = x^{-\frac{1}{3}}\lambda^{-\frac{9}{2n+1}}\,(1-m)\sqrt{\frac{\alpha}{\alpha - m}}\,\gamma_3 \tag{14}$$

. . . .

$$c_{2r} = x^{-\frac{1}{3}}\lambda^{-\frac{4r^2}{2n+1}}\,\frac{m\alpha}{\alpha - m}\,\frac{N_{2r}}{D_{2r}}\,\gamma_{2r} \tag{15}$$

$$c_{2r+1} = x^{-\frac{1}{3}}\lambda^{-\frac{(2r+1)^2}{2n+1}}\, m\sqrt{\frac{\alpha}{\alpha - m}}\cdot\frac{N_{2r+1}}{D_{2r+1}}\,\gamma_{2r+1} \tag{16}$$

we find that

$$c_r = \frac{\Theta\, rf\mathrm{K}'}{\Theta 0} \quad \text{or} \quad \frac{\mathrm{H} rf\mathrm{K}'}{\mathrm{HK}'} \tag{17}$$

or one of its linear transformations; and $\sqrt{\frac{\alpha}{\alpha - m}}$ is always rational in α and m, in consequence of the relation

$$s\,(\omega) - s\,(nv) = s\,(\omega) - s\,(n+1)\,v \tag{18}$$

The function c is the one that should be tabulated numerically, as

$$\operatorname{sn}u = \frac{\mathrm{H}u}{\mathrm{HK}}\Big/\frac{\Theta u}{\Theta\mathrm{K}}, \quad \operatorname{cn}u = \frac{\mathrm{H}(\mathrm{K}-u)}{\mathrm{HK}}\Big/\frac{\Theta u}{\Theta 0}, \quad \operatorname{dn}u = \frac{\Theta(\mathrm{K}-u)}{\Theta\mathrm{K}}\Big/\frac{\Theta u}{\Theta 0} \tag{19}$$

Also

$$\frac{\mathrm{P}(v)}{\sqrt{(a^2-b^2)}} = \operatorname{zs} f\mathrm{K}' \tag{20}$$

or one of its transformations, depending on the region.

29. $\mu = 6$, $f = \frac{1}{3}$ or $\frac{2}{3}$.

Derive from $\mu = 3$ in (3) § 6, by putting $c = a^2(2a-1)$, so that

$$\begin{aligned}\mathrm{I}(v) &= \frac{2}{3}\cos^{-1}\frac{\sqrt{(t+a)}\sqrt{\{2t^2-(2a-1)t+2a^2-a\}}}{2t^{\frac{3}{2}}}\\ &= \frac{2}{3}\sin^{-1}\frac{\sqrt{(t-a)}\sqrt{\{2t^2+(2a-1)t+2a^2-a\}}}{2t^{\frac{3}{2}}}\\ &= \int_a^t \frac{\frac{1}{3}t^2+2a^3-a^2}{t}\cdot\frac{dt}{\sqrt{\{4t^6-(t^2+2a^3-a^2)^2\}}}\end{aligned} \tag{1}$$

Next put

$$a = \frac{1}{2}\frac{b^2-1}{b^2+3} \tag{2}$$

$$t_1 = \frac{b+1}{b^2+3}, \quad t_2 = \frac{1}{2}\frac{b^2-1}{b^2+3}, \quad t_3 = \frac{b-1}{b^2+3} \tag{3}$$

$$\kappa^2 = \frac{(b-1)^3(b+3)}{16b}, \quad \kappa'^2 = \frac{(b+1)^3(-b+3)}{16b} \tag{4}$$

$$\mathrm{L}(2)^{24} = \frac{4096b^2}{(b^2-1)^3(b^2-9)} \tag{5}$$

Then from (8), § 23, in the region $3 > b > 1$,

$$\operatorname{cn}\tfrac{2}{3}\mathrm{K}' = \frac{b-1}{b+1}, \quad \operatorname{dn}\tfrac{2}{3}\mathrm{K}' = \frac{b-1}{2}, \quad \operatorname{sn}\tfrac{1}{3}\mathrm{K}' = \frac{2}{b+1} \tag{6}$$

so that there results the well-known relation

$$\operatorname{cn}\tfrac{2}{3}\mathrm{K}' + \operatorname{sn}\tfrac{1}{3}\mathrm{K}' = 1 \tag{7}$$

and in LEGENDRE, 1, chap. vi., p. 27,

$$x = \sin\phi = \operatorname{sn}\tfrac{1}{3}\mathrm{K} = \frac{2}{b+1}, \quad \kappa^2 = \frac{1-2x}{-2x^3+x^4} \tag{8}$$

The following table shows the six linear transformations of the division-values for $\mu = 6$, obtained in accordance with the preceding theory by the substitutions

$$\left(b, \quad \frac{b+3}{b-1}, \quad -b\right) \tag{9}$$

the accents are omitted from K by changing to the complementary modulus.

The numerical value $b = \sqrt{3}$ gives a modular angle 75° ; while $b = \sqrt{5}$ corresponds to the functions required for $\mu = 10$.

Region.	A. $\infty > b > 3.$	B. $3 > b > 1.$	C. $1 > b > 0.$	D. $0 > b > -1.$	E. $-1 > b > -3.$	F. $-3 > b > -\infty$
$\frac{(b+1)^3(-b+3)}{16b}$	$-\frac{\kappa'^2}{\kappa^2}$	κ^2	$-\frac{\kappa^2}{\kappa'^2}$	$\frac{1}{\kappa'^2}$	κ'^2	$-\frac{\kappa'^2}{\kappa^2}$
$\frac{(b-1)^3(b+3)}{16b}$	$\frac{1}{\kappa^2}$	κ'^2	$\frac{1}{\kappa'^2}$	$-\frac{\kappa^2}{\kappa'^2}$	κ^2	$\frac{1}{\kappa^2}$
$\frac{2}{b+1}$	$\mathrm{cn}\,\frac{2}{3}\mathrm{K}$	$\mathrm{sn}\,\frac{1}{3}\mathrm{K}$	$\mathrm{ns}\,\frac{1}{3}\mathrm{K}$	$\mathrm{nc}\,\frac{2}{3}\mathrm{K}$	$-\mathrm{nd}\,\frac{2}{3}\mathrm{K}$	$-\mathrm{dn}\,\frac{2}{3}\mathrm{K}$
$\frac{b-1}{b+1}$	$\mathrm{sn}\,\frac{1}{3}\mathrm{K}$	$\mathrm{cn}\,\frac{2}{3}\mathrm{K}$	$-\mathrm{dn}\,\frac{2}{3}\mathrm{K}$	$-\mathrm{nd}\,\frac{2}{3}\mathrm{K}$	$\mathrm{nc}\,\frac{2}{3}\mathrm{K}$	$\mathrm{ns}\,\frac{1}{3}\mathrm{K}$
$\frac{b-1}{2}$	$\mathrm{nd}\,\frac{2}{3}\mathrm{K}$	$\mathrm{dn}\,\frac{2}{3}\mathrm{K}$	$-\mathrm{cn}\,\frac{2}{3}\mathrm{K}$	$-\mathrm{sn}\,\frac{1}{3}\mathrm{K}$	$-\mathrm{ns}\,\frac{1}{3}\mathrm{K}$	$-\mathrm{nc}\,\frac{2}{3}\mathrm{K}$
$\frac{-b^2+9}{12\sqrt{b}}$	$-\frac{1}{\kappa'}\mathrm{zn}\,\frac{2}{3}\mathrm{K}$	$\mathrm{zn}\,\frac{1}{3}\mathrm{K}$	$\frac{1}{\kappa}\mathrm{zc}\,\frac{2}{3}\mathrm{K}$	$\frac{i}{\kappa}\mathrm{zc}\,\frac{2}{3}\mathrm{K}$	$i\,\mathrm{zn}\,\frac{1}{3}\mathrm{K}$	$-\frac{i}{\kappa'}\mathrm{zn}\,\frac{2}{3}\mathrm{K}$
$\frac{(b+3)\sqrt{b}}{6}$	$\frac{1}{\kappa'}\mathrm{zs}\,\frac{1}{3}\mathrm{K}$	$\mathrm{zs}\,\frac{1}{3}\mathrm{K}$	$\frac{1}{\kappa}\mathrm{zn}\,\frac{1}{3}\mathrm{K}$	$\frac{i}{\kappa}\mathrm{zn}\,\frac{2}{3}\mathrm{K}$	$i\,\mathrm{zn}\,\frac{2}{3}\mathrm{K}$	$\frac{i}{\kappa'}\mathrm{zn}\,\frac{1}{3}\mathrm{K}$
$\frac{(-6+3)\sqrt{b})}{6}$	$-\frac{1}{\kappa'}\mathrm{zn}\,\frac{1}{3}\mathrm{K}$	$\mathrm{zn}\,\frac{2}{3}\mathrm{K}$	$\frac{1}{\kappa}\mathrm{zn}\,\frac{2}{3}\mathrm{K}$	$\frac{i}{\kappa}\mathrm{zn}\,\frac{1}{3}\mathrm{K}$	$i\,\mathrm{zs}\,\frac{1}{3}\mathrm{K}$	$\frac{i}{\kappa'}\mathrm{zs}\,\frac{1}{3}\mathrm{K}$
$\frac{(b^2-1)^{\frac{1}{3}}}{2}$	$\frac{\Theta\frac{2}{3}\mathrm{K}}{\Theta 0}$	$\frac{\Theta\frac{1}{3}\mathrm{K}}{\Theta\mathrm{K}}$	$-\frac{\mathrm{H}\frac{1}{3}\mathrm{K}}{\mathrm{HK}}$	$-\frac{\mathrm{H}\frac{1}{3}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\frac{1}{3}\mathrm{K}}{\Theta\mathrm{K}}$	$\frac{\Theta\frac{2}{3}\mathrm{K}}{\Theta 0}$
$\frac{(b+1)^{\frac{1}{3}}}{(b-1)^{\frac{2}{3}}}$	$\frac{\Theta\frac{1}{3}\mathrm{K}}{\Theta\mathrm{K}}$	$\frac{\Theta\frac{2}{3}\mathrm{K}}{\Theta 0}$	$\frac{\Theta\frac{2}{3}\mathrm{K}}{\Theta 0}$	$\frac{\Theta\frac{1}{3}\mathrm{K}}{\Theta\mathrm{K}}$	$-\frac{\mathrm{H}\frac{1}{3}\mathrm{K}}{\mathrm{HK}}$	$-\frac{\mathrm{H}\frac{1}{3}\mathrm{K}}{\mathrm{HK}}$
$\frac{(b-1)^{\frac{1}{3}}}{(b+1)^{\frac{2}{3}}}$	$\frac{\mathrm{H}\frac{1}{3}\mathrm{K}}{\mathrm{HK}}$	$\frac{\mathrm{H}\frac{1}{3}\mathrm{K}}{\mathrm{HK}}$	$-\frac{\Theta\frac{1}{3}\mathrm{K}}{\Theta\mathrm{K}}$	$-\frac{\Theta\frac{2}{3}\mathrm{K}}{\Theta 0}$	$-\frac{\Theta\frac{2}{3}\mathrm{K}}{\Theta 0}$	$-\frac{\Theta\frac{1}{3}\mathrm{K}}{\Theta\mathrm{K}}$
Test by b	$2\sqrt{3}+3$	$\sqrt{3}$	$2\sqrt{3}-3$	$-2\sqrt{3}+3$	$-\sqrt{3}$	$-2\sqrt{3}-3$
or b	$\sqrt{5}+2$	$\sqrt{5}$	$\sqrt{5}-2$	$-\sqrt{5}+2$	$-\sqrt{5}$	$-\sqrt{5}-2$

30. $\mu = 10$, $f = \frac{1,\,2,\,3,\,4}{5}$.

$$D_5 = 0, \quad \alpha = \frac{m(1-m)^2}{1-2m}, \quad \alpha - m = \frac{m^3}{1-2m} \tag{1}$$

$$x = y = -\frac{(1-m)^2(1-2m)}{m^2} \tag{2}$$

To identify with the results in L.M.S., 27, put

$$2m - 1 = \frac{1}{c}, \quad m = \frac{c+1}{2c}, \quad 1 - m = \frac{c-1}{2c} \tag{3}$$

then

$$T_1, T_2 = \left[t \pm \frac{(c-1)^2}{2c(c+1)}\right]\left[2t^2 \pm \frac{4c}{(c+1)^2}t + \frac{2(c-1)^2}{c(c+1)^3}\right] \tag{4}$$

and from (9), § 7,

$$\begin{aligned} I(4v) &= \frac{2}{5}\cos^{-1}\frac{\left\{t - \frac{(c-1)^2}{c(c+1)^2}\right\}\sqrt{T_1}}{2t^{\frac{5}{2}}} \\ &= \frac{2}{5}\sin^{-1}\frac{\left\{t + \frac{(c-1)^2}{c(c+1)^2}\right\}\sqrt{T_2}}{2t^{\frac{5}{2}}} \\ &= \int \frac{\frac{c^3 - c^2 + 7c - 3}{10c(c+1)^2}t^2 - \frac{(c-1)^4}{2c^2(c+1)^4}}{t^2}\frac{dt^2}{\sqrt{T_1T_2}} \end{aligned} \tag{5}$$

Put

$$t = \frac{(c-1)^2}{2c(c+1)}y, \quad \text{and} \quad y^2 = \frac{s(v) - s}{s(\omega) - s} \tag{6}$$

with

$$s(v) = 8c(c+1)^2(c-1), \quad s(\omega) = (c+1)^2(c-1)^4 \tag{7}$$

and we obtain $I(v)$ in the form employed in L.M.S., 27, p. 601, derived by putting $N_5 = 0$, with

$$P(v) = \tfrac{1}{5}(c+3)(-c^2+4c+1), \quad Q(v) = 4c(c+1)^2(c-1)(-c^2+4c+1) \tag{8}$$

and in Region II., $f = \frac{1}{5}$ or $\frac{3}{5}$,

$$s_1 = 4(c^2 + \sqrt{C})^2, \quad s_2 = (c+1)^2(c-1)^4, \quad s_3 = 4(c^2 - \sqrt{C})^2 \tag{9}$$

where

$$C = c^3 + c^2 - c \tag{10}$$

and

$$L(2)^{24} = \frac{4096c^4C}{(c^2-1)^5(c^2-4c-1)} \tag{11}$$

This c is connected with FRICKE's τ and t ('Diss.,' Leipzig, 1886) by the relation

$$c = \frac{\frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}-1}{2}\right)^2}{\tau - \frac{\sqrt{5}+1}{2\sqrt{5}}}, \quad t = \frac{-c^2+4c+1}{5c} \tag{12}$$

The following table, analogous to the preceding one for $\mu = 6$, gives the division-values for $\mu = 10$, the six transformations being derived from the substitution

$$\left\{c, \quad \left[\frac{\sqrt{(c^2+c-1)} \pm \sqrt{c}}{c-1}\right]^2, \quad -\frac{1}{c}\right\} \tag{13};$$

and it may be compared with the corresponding results given by F. MÜLLER, 'Archiv der Mathematik und Physik,' p. 161, 1884, ROHDE, 'Archiv,' 1886, p. 138, and by W. GÖRING, 'Math. Ann.,' 7, p. 311, 1874, working on GAUSS's unpublished memoirs.

For the application of this integral to a case of motion of the top, consult the 'Annals of Mathematics,' vol. 5, 1903.

The numerical value $c = 2$ will serve for verification; this corresponds to the case of ten cusps at intervals of $\frac{1}{5}\pi$ in the associated motion of the top (L.M.S., 27, p. 602); and $c = 2(\sin 36° \pm \sin 18°)$ will make $K'/K = \sqrt{5}$ or 1, cases of Complex Multiplication.

Region.	A. $\infty > c > \sqrt{5}+2.$	B. $\sqrt{5}+2 > c > 1.$
$\dfrac{(2c-1+\sqrt{C})(2c+1-\sqrt{C})(\sqrt{C}+1)^2}{16c^2\sqrt{C}}$	$\dfrac{1}{\kappa^2}$	κ^2
$\dfrac{(2c+1+\sqrt{C})(-2c+1+\sqrt{C})(\sqrt{C}-1)^2}{16c^2\sqrt{C}}$	$-\dfrac{\kappa'^2}{\kappa^2}$	κ'^2
$\dfrac{2c}{c^2+\sqrt{C}}$	$\mathrm{cn}\,\frac{4}{5}\mathrm{K}$	$\mathrm{sn}\,\frac{1}{5}\mathrm{K}$
$\dfrac{2c}{\sqrt{C}+1}$	$\mathrm{cn}\,\frac{2}{5}\mathrm{K}$	$\mathrm{sn}\,\frac{3}{5}\mathrm{K}$
$\dfrac{\sqrt{C}-1}{\sqrt{C}+1}$	$\mathrm{sn}\,\frac{3}{5}\mathrm{K}$	$\mathrm{cn}\,\frac{2}{5}\mathrm{K}$
$\dfrac{\sqrt{C}-1}{2c}$	$\mathrm{nd}\,\frac{2}{5}\mathrm{K}$	$\mathrm{dn}\,\frac{2}{5}\mathrm{K}$
$\dfrac{c^2-\sqrt{C}}{c^2+\sqrt{C}}$	$\mathrm{sn}\,\frac{1}{5}\mathrm{K}$	$\mathrm{cn}\,\frac{4}{5}\mathrm{K}$
$\dfrac{c^2-\sqrt{C}}{2c}$	$\mathrm{nd}\,\frac{4}{5}\mathrm{K}$	$\mathrm{dn}\,\frac{4}{5}\mathrm{K}$
$\dfrac{(c+3)(-c^2+4c+1)}{20c\sqrt[4]{C}}$	$-\dfrac{1}{\kappa'}\,\mathrm{zn}\,\frac{4}{5}\mathrm{K}$	$\mathrm{zn}\,\frac{1}{5}\mathrm{K}$
$\dfrac{(3c-1)(-c^2+4c+1)}{20c\sqrt[4]{C}}$	$-\dfrac{1}{\kappa'}\,\mathrm{zn}\,\frac{2}{5}\mathrm{K}$	$\mathrm{zn}\,\frac{3}{5}\mathrm{K}$
$\dfrac{c^3-c^2+7c-3}{20c\sqrt[4]{C}}$	$\dfrac{1}{\kappa'}\,\mathrm{zs}\,\frac{4}{5}\mathrm{K}$	$\mathrm{zs}\,\frac{4}{5}\mathrm{K}$
$\dfrac{3c^3+7c^2+c+1}{20c\sqrt[4]{C}}$	$\dfrac{1}{\kappa'}\,\mathrm{zs}\,\frac{2}{5}\mathrm{K}$	$\mathrm{zs}\,\frac{2}{5}\mathrm{K}$
$\dfrac{(c+1)^{\frac{1}{5}}(c-1)^{\frac{4}{5}}}{2c^{\frac{2}{5}}}$	$\dfrac{\Theta\,\frac{4}{5}\mathrm{K}}{\Theta 0}$	$\dfrac{\Theta\,\frac{1}{5}\mathrm{K}}{\Theta\mathrm{K}}$
$\dfrac{(c+1)^{\frac{4}{5}}(c-1)^{\frac{1}{5}}}{2c^{\frac{3}{5}}}$	$\dfrac{\Theta\,\frac{2}{5}\mathrm{K}}{\Theta 0}$	$\dfrac{\Theta\,\frac{3}{5}\mathrm{K}}{\Theta\mathrm{K}}$

C. $1 > c > \frac{\sqrt{5}-1}{2}$.	D. $0 > c > -\sqrt{5}+2$.	E. $-\sqrt{5}+2 > c > -1$.	F. $-1 > c > -\frac{\sqrt{5}+1}{2}$.
$-\frac{\kappa^2}{\kappa'^2}$	$\frac{1}{\kappa'^2}$	κ'^2	$-\frac{\kappa'^2}{\kappa^2}$
$\frac{1}{\kappa'^2}$	$-\frac{\kappa^2}{\kappa'^2}$	κ^2	$\frac{1}{\kappa^2}$
$\mathrm{ns}\,\frac{1}{5}\mathrm{K}$	$\mathrm{dn}\,\frac{2}{5}\mathrm{K}$	$\mathrm{nd}\,\frac{2}{5}\mathrm{K}$	$-\mathrm{nc}\,\frac{2}{5}\mathrm{K}$
$\mathrm{ns}\,\frac{3}{5}\mathrm{K}$	$-\mathrm{dn}\,\frac{4}{5}\mathrm{K}$	$-\mathrm{nd}\,\frac{4}{5}\mathrm{K}$	$-\mathrm{nc}\,\frac{4}{5}\mathrm{K}$
$-\mathrm{dn}\,\frac{2}{5}\mathrm{K}$	$-\mathrm{ns}\,\frac{1}{5}\mathrm{K}$	$-\mathrm{nc}\,\frac{4}{5}\mathrm{K}$	$-\mathrm{nd}\,\frac{4}{5}\mathrm{K}$
$-\mathrm{cn}\,\frac{2}{5}\mathrm{K}$	$\mathrm{nc}\,\frac{4}{5}\mathrm{K}$	$\mathrm{ns}\,\frac{1}{5}\mathrm{K}$	$\mathrm{sn}\,\frac{1}{5}\mathrm{K}$
$\mathrm{dn}\,\frac{4}{5}\mathrm{K}$	$-\mathrm{ns}\,\frac{3}{5}\mathrm{K}$	$-\mathrm{nc}\,\frac{2}{5}\mathrm{K}$	$\mathrm{nd}\,\frac{2}{5}\mathrm{K}$
$\mathrm{cn}\,\frac{4}{5}\mathrm{K}$	$-\mathrm{nc}\,\frac{2}{5}\mathrm{K}$	$-\mathrm{ns}\,\frac{3}{5}\mathrm{K}$	$-\mathrm{sn}\,\frac{3}{5}\mathrm{K}$
$-\frac{1}{\kappa}\,\mathrm{zs}\,\frac{1}{5}\mathrm{K}$	$\frac{i}{\kappa'}\,\mathrm{zn}\,\frac{2}{5}\mathrm{K}$	$i\,\mathrm{zn}\,\frac{3}{5}\mathrm{K}$	$\frac{i}{\kappa}\,\mathrm{zc}\,\frac{2}{5}\mathrm{K}$
$-\frac{1}{\kappa}\,\mathrm{zs}\,\frac{3}{5}\mathrm{K}$	$\frac{i}{\kappa}\,\mathrm{zn}\,\frac{4}{5}\mathrm{K}$	$i\,\mathrm{zn}\,\frac{1}{5}\mathrm{K}$	$\frac{i}{\kappa}\,\mathrm{zc}\,\frac{4}{5}\mathrm{K}$
$\frac{1}{\kappa}\,\mathrm{zs}\,\frac{4}{5}\mathrm{K}$	$\frac{i}{\kappa}\,\mathrm{zs}\,\frac{2}{5}\mathrm{K}$	$i\,\mathrm{zs}\,\frac{2}{5}\mathrm{K}$	$\frac{i}{\kappa}\,\mathrm{zs}\,\frac{2}{5}\mathrm{K}$
$\frac{1}{\kappa}\,\mathrm{zs}\,\frac{2}{5}\mathrm{K}$	$\frac{i}{\kappa}\,\mathrm{zs}\,\frac{4}{5}\mathrm{K}$	$-i\,\mathrm{zs}\,\frac{4}{5}\mathrm{K}$	$\frac{i}{\kappa}\,\mathrm{zs}\,\frac{4}{5}\mathrm{K}$
$\frac{\mathrm{H}\,\frac{1}{5}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{2}{5}\mathrm{K}}{\Theta 0}$	$\frac{\Theta\,\frac{3}{5}\mathrm{K}}{\Theta\mathrm{K}}$	$-\frac{\mathrm{H}\,\frac{3}{5}\mathrm{K}}{\mathrm{HK}}$
$-\frac{\mathrm{H}\,\frac{3}{5}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{4}{5}\mathrm{K}}{\Theta 0}$	$\frac{\Theta\,\frac{1}{5}\mathrm{K}}{\Theta\mathrm{K}}$	$\frac{\mathrm{H}\,\frac{1}{5}\mathrm{K}}{\mathrm{HK}}$

31. $\mu = 7, 14$; $f = \frac{1, 2, 3, 4, 5, 6}{7}$;

$$D_7 = (1 - 2m)^3 \alpha^2 - m(1 - 2m)(1 - 3m)\alpha - m^4(1 - m)^2 = 0 \qquad (1);$$

and on putting

$$1 - 2m = p, \quad (1 - 2m)\alpha = m(1 - m)\beta, \quad 2\beta = 1 + \frac{1}{q} \qquad (2),$$

the relation becomes

$$p^2q^2 + pq - p - q = 0 \qquad (3),$$

an elliptic-function addition equation, connecting

$$p = p(u) \quad \text{and} \quad q = p(u + \tfrac{2}{3}\omega) \qquad (4),$$

$$u = \int \frac{dp}{\sqrt{P}}, \quad P = 4p^3 + (p - 1)^2 = (p + 1)(4p^2 - 3p + 1) \qquad (5),$$

and

$$p(\tfrac{2}{3}\omega) = 0, \quad p(\tfrac{1}{3}\omega) = 1 \qquad (6).$$

Thence

$$\alpha = \frac{(p^2 - 1)(-3p + 1 + \sqrt{P})}{16p^2} \qquad (7),$$

$$\frac{\alpha}{\alpha - m} = \frac{(p + 1)\{p\sqrt{(p + 1)} - \sqrt{(4p^2 - 3p + 1)}\}^2}{(p - 1)^4} \qquad (8),$$

$$L(2)^{24} = \frac{p(p - 1)\{7p(p^2 - 1) - (p^2 + p + 2)\sqrt{P}\}}{2(p^3 - 9p^2 - p + 1)} \qquad (9).$$

The connexion with KIEPERT's results in 'Math. Ann.,' 32, p. 87, has been given in L.M.S., 27, p. 438; here

$$\xi = 8\xi_4 = \frac{(p + 1)(p - 1)^2}{p^2} \qquad (10),$$

$$\frac{d\xi}{\sqrt{(4\xi^3 + 13\xi^2 + 32\xi)}} = \frac{dp}{\sqrt{P}} = du \qquad (11),$$

a transformation of the Third Order; and

$$\int_{-1}^{0} \frac{dp}{\sqrt{P}} = \int_0^1 = \int_1^\infty = \int_0^\infty \frac{d\xi}{\xi'} = \tfrac{1}{3}\omega \qquad (12),$$

suppose, where ω is the real half-period of the elliptic function $p = p(u)$ of the elliptic argument u, and now

$$\xi_1 = \xi(u - \tfrac{1}{9}\omega), \quad \frac{1}{\xi_4} = \xi(u - \tfrac{1}{3}\omega) \qquad (13),$$

$$\xi\frac{2r\omega}{9} = 1, \quad p^3 - 2p^2 - p + 1 = 0 \qquad (14),$$

the roots of which are

$$p = p\left(\frac{2r\omega}{9}\right) = \tfrac{1}{2}\sec\frac{1,\ 3,\ 5}{7}\pi \qquad (15),$$

$$\xi\frac{r\omega}{9} = 8, \quad p^3 - 9p^2 - p + 1 = 0 \qquad (16),$$

the roots of which are derived from the preceding, by the substitution

$$\left(p + 1, \quad \frac{2}{p+1}\right) \qquad (17).$$

Thence the contour of the period parallelogram is divided into segments of $\frac{1}{9}\omega$, and a similar table of division-values for $\mu = 14$ can be constructed for the corresponding regions of p and compared with the results of GÖRING and MÜLLER; this must be reserved for the present.

The associated τ parameters of GIERSTER ('Math. Ann.,' 14) have simple numerical values:

$$\tau_{18} = -1, \quad \tau_{12} = 2, \quad \tau_9 = \tfrac{4}{3}, \quad \tau_6 = -\tfrac{7}{2}, \quad \tau_4 = -2, \quad \tau_3 = \tfrac{28}{27}, \quad \tau_2 = \tfrac{7}{128} \qquad (18).$$

The parameter z, employed in § 9 (2), is given in terms of p by the relation

$$z = \frac{y - x}{y} = \frac{s(3v) - s(2v)}{s(3v) - s(v)}$$

$$= \frac{(1-m)^2 - \dfrac{m^2\alpha}{\alpha - m}}{1 - 2m} = \frac{(p+1)(-3p + 1 + \sqrt{\mathrm{P}})}{2(p-1)^2} \qquad (19);$$

thence h_1, h_2 in § 9 (6) can be determined, and T_1, T_2 in factors, as required for the Second Stage of $\mu = 14$, in terms of p and $\sqrt{\mathrm{P}}$.

Changing from the variable t to y, in § 28 (1), we find

$$\mathrm{I} = \int \frac{\dfrac{2\mathrm{P}(v)}{\mathrm{M}} y^2 - \mathrm{Q}}{y} \frac{dy}{\sqrt{(\mathrm{Y}_1\mathrm{Y}_2)}}$$

$$= \frac{2}{7}\cos^{-1}\frac{y^2 + \mathrm{A}_1 y + \mathrm{A}_2}{y^{\frac{3}{2}}}\sqrt{\tfrac{1}{2}\mathrm{Y}_2}$$

$$= \frac{2}{7}\sin^{-1}\frac{y^2 - \mathrm{A}_1 y + \mathrm{A}_2}{y^{\frac{3}{2}}}\sqrt{\tfrac{1}{2}\mathrm{Y}_1} \qquad (20),$$

$$\mathrm{Y}_2 = (y-1)(y^2 - \mathrm{C}_1 y + \mathrm{C}_2)$$
$$\mathrm{Y}_1 = (y+1)(y^2 + \mathrm{C}_1 y + \mathrm{C}_2) \qquad (21),$$

$$\mathrm{C}_1 = -\frac{1}{2\alpha} = -\frac{2p(3p - 1 + \sqrt{\mathrm{P}})}{(p+1)(p-1)^3}$$

$$\mathrm{C}_2 = -p\mathrm{C}_1 = \frac{2p^2(3p - 1 + \sqrt{\mathrm{P}})}{(p+1)(p-1)^3} \qquad (22),$$

$$A_1 = -\frac{\{\sqrt{(4p^2 - 3p + 1)} - \sqrt{(p+1)}\}^2}{2(p^2 - 1)}$$

$$A_2 = \frac{2pA_1}{p - 1} \qquad (23),$$

$$\frac{P(v)}{M} = \frac{-5p^4 + 12p^3 - 4p^2 + 2p - 1 + 2(p^2 - p + 1)\sqrt{P}}{14(p+1)(p-1)^3} \qquad (24),$$

$$Q = \frac{1 - 2m}{2\alpha} = \frac{2p^2(3p - 1 + \sqrt{P})}{(p+1)(p-1)^3} \qquad (25).$$

Test by $$p = \sqrt{2} - 1 = p\tfrac{1}{2}\omega, \quad \frac{K'}{K} = \sqrt{14}.$$

For the elliptic-section values the region may be selected in which, in accordance with § 28,

$$\kappa^2 \mathrm{tn}^4 \tfrac{2}{7} K' = a_1 = \frac{(p+1)^{\frac{3}{2}}\{(p^3 - 7p^2 + 3p - 1)\sqrt{(p+1)} + 4p\sqrt{(4p^2 - 3p + 1)}\}}{(p-1)^5} \qquad (26),$$

$$\kappa^2 \mathrm{tn}^4 \tfrac{5}{7} K' = \frac{1}{a_1} = \frac{(p-1)\{(p^3 - 7p^2 + 3p - 1)\sqrt{(p+1)} - 4p\sqrt{(4p^2 - 3p + 1)}\}}{(p^3 - 9p^2 - p + 1)(p+1)^{\frac{3}{2}}} \qquad (27),$$

$$\kappa^2 \mathrm{tn}^4 \tfrac{4}{7} K' = a_2 = \frac{(p+1)^{\frac{1}{2}}\{(p^2 + 2p - 1)\sqrt{(p+1)} + 2p\sqrt{(4p^2 - 3p + 1)}\}}{(p-1)^3} \qquad (28),$$

$$\kappa^2 \mathrm{tn}^4 \tfrac{3}{7} K' = \frac{1}{a_2} = \frac{(p-1)\{(p^2 + 2p - 1)\sqrt{(p+1)} - 2p\sqrt{(4p^2 - 3p + 1)}\}}{(p^3 - 9p^2 - p + 1)(p+1)^{\frac{1}{2}}} \qquad (29),$$

$$\kappa^2 \mathrm{tn}^4 \tfrac{6}{7} K' = a_3 = \frac{(p^3 - 7p^2 + 3p - 1)\sqrt{(p+1)} + 4p\sqrt{(4p^2 - 3p + 1)}}{(p-1)^3(p+1)^{\frac{1}{2}}} \qquad (30),$$

$$\kappa^2 \mathrm{tn}^4 \tfrac{1}{7} K' = \frac{1}{a_3} = \frac{(p+1)^{\frac{1}{2}}\{(p^3 - 7p^2 + 3p - 1)\sqrt{(p+1)} - 4p\sqrt{(4p^2 - 3p + 1)}\}}{(p^3 - 9p^2 - p + 1)(p-1)} \qquad (31).$$

With

$$\gamma_7 = 0, \quad \lambda = \frac{\gamma_4}{\gamma_3} = \frac{y}{x^{\frac{1}{3}}}, \quad x^{\frac{1}{3}}\lambda^{\frac{1}{7}} = x^{\frac{2}{7}}y^{\frac{1}{7}} \qquad (32),$$

$$\left(\frac{\Theta\, \frac{2}{7} K'}{\Theta 0}\right)^7 = c_1{}^7 = \frac{m\alpha^2}{1 - 2m}\left(\frac{\alpha - m}{\alpha}\right)^{\frac{3}{2}} \qquad (33),$$

$$\left(\frac{\Theta\, \frac{4}{7} K'}{\Theta 0}\right)^7 = c_2{}^7 = \frac{m^4\alpha}{(1 - 2m)^7}\left(\frac{\alpha}{\alpha - m}\right) \qquad (34),$$

$$\left(\frac{\Theta\, \frac{6}{7} K'}{\Theta 0}\right)^7 = c_3{}^7 = \frac{m^9(1 - m)^7}{(1 - 2m)^9\alpha^3}\left(\frac{\alpha}{\alpha - m}\right)^{\frac{1}{2}} \qquad (35).$$

32. $\mu = 18$, $f = \frac{1, 2, 4, 5, 7, 8}{9}$.

The relation $D_9 = 0$ becomes by the substitutions

$$\alpha = \frac{m(1-m)\beta}{1-2m}, \quad 1 - 2m = p, \quad 2\beta - 1 = \epsilon \tag{1},$$

the same as those employed with D_{11} in § 22, a new relation

$$-p^3 + \epsilon^2(\epsilon - 2)p^2 + \epsilon(\epsilon^2 - \epsilon + 1)p + \epsilon^3 = 0. \tag{2},$$

representing a C_5 in (p, ϵ); and putting

$$p = -\frac{1}{x+1}, \quad \epsilon = \frac{1}{y+1} \tag{3},$$

this C_5 reduces to a C_4 in (x, y) with deficiency $p = 2$,

$$y^3 - (x^2 + 2x - 2)y^2 - (x+4)xy + x^2(x+2) = 0 \tag{4},$$

and this with $y = (q+1)x$, as in L.M.S., 27, p. 463, becomes

$$(q+1)^2 x^2 - (q^3 + q^2 - 2q - 1)x - 2q^2 = 0 \tag{5},$$

$$x = \frac{q^3 + q^2 - 2q - 1 + \sqrt{Q}}{2(q+1)^2} \tag{6},$$

$$x + 1 = \frac{q^3 + 3q^2 + 2q + 1 + \sqrt{Q}}{2(q+1)^2} \tag{7},$$

$$Q = q^6 + 2q^5 + 5q^4 + 10q^3 + 10q^2 + 4q + 1 \tag{8},$$

$$\frac{1}{x+1} = \frac{q^3 + 3q^2 + 2q + 1 - \sqrt{Q}}{2q^3} = -p \tag{9},$$

$$y = \frac{q^3 + q^2 - 2q - 1 + \sqrt{Q}}{2(q+1)} \tag{10},$$

$$\frac{1}{y+1} = \frac{-q^3 - q^2 - 0 - 1 + \sqrt{Q}}{2q(q^2 + q + 1)} = \epsilon \tag{11}.$$

Now the p employed in (2), § 10, distinguished here as p_9, is given by

$$x = y(1 - z), \quad z - y = \frac{z^2}{p} \tag{12},$$

returning for a moment to the original x and y of § 10, so that

$$p_9 = \frac{z^2}{z-y} = \frac{(y-x)^2}{y(y-x-y^2)}$$

$$= \frac{s(2v)-s(v)}{s(5v)-s(v)} = \frac{s(2v)-s(v)}{s(4v)-s(v)}$$

$$= \frac{\dfrac{m^2\alpha}{\alpha-m} - \dfrac{m^2\alpha^2}{(\alpha-m)^2}}{\dfrac{m^2\alpha}{\alpha-m} - \dfrac{m^2\{(1-2m)\alpha - m(1-m)\}^2}{(1-2m)^2(\alpha-m)^2}}$$

$$= \frac{-(1-2m)^2\alpha}{(1-2m)\alpha - m(1-m)^2} = \frac{-(1-2m)\beta}{\beta-1+m}$$

$$= \frac{-p(\epsilon+1)}{\epsilon-p} \quad (13);$$

and

$$\epsilon - p = \frac{(q+1)(3q^3+3q^2+2q+1-\sqrt{Q})}{2q^3(q^2+q+1)} \quad (14),$$

$$\frac{1}{\epsilon-p} = \frac{3q^3+3q^2+2q+1+\sqrt{Q}}{4(q+1)^2} \quad (15),$$

$$\epsilon+1 = \frac{q^3+q^2+2q-1+\sqrt{Q}}{2q(q^2+q+1)} = 2\beta \quad (16),$$

$$\frac{\epsilon+1}{\epsilon-p} = \frac{q^3+q^2+2q+1+\sqrt{Q}}{2(q+1)^2} \quad (17),$$

$$p_9 = \frac{-p(\epsilon+1)}{\epsilon-p} = \frac{q^3+q^2+0-1+\sqrt{Q}}{2q(q+1)^2} \quad (18).$$

This value of p_9, employed in the previous equations (7), § 10, for $\mu = 9$ will lead to the functions of the Second Stage for $\mu = 18$; the result, in a modified form, has been given already in the 'Archiv der Mathematik und Physik,' 3, Reihe, 1, p. 74.

Put

$$q = -\frac{1}{a+1}, \quad Q = \frac{A}{(a+1)^6}, \quad A = a^6+2+5+10+10+4+1 \quad (19),$$

to agree with the results in the 'Archiv,' and in the modified form of § 28

$$I = \int \frac{Py^2-Q}{y}\frac{dy}{\sqrt{Y_1Y_2}}$$

$$= \frac{2}{9}\cos^{-1}\frac{y^3+A_1y^2+A_2y+A_3}{y^{\frac{9}{2}}}\sqrt{\tfrac{1}{2}Y_1}$$

$$= \frac{2}{9}\sin^{-1}\frac{y^3-A_1y^2+A_2y-A_3}{y^{\frac{9}{2}}}\sqrt{\tfrac{1}{2}Y_2} \quad (20),$$

$$Y_1 = (y + 1)(y^2 + C_1 y + C_2)$$

$$Y_2 = (y - 1)(y^2 - C_1 y + C_2) \qquad (21),$$

$$C_1 = -\frac{1}{2\alpha}$$

$$C_2 = \frac{1 - 2m}{2\alpha} = Q \qquad (22),$$

$$A_1 = \frac{2\alpha^5 + 6 + 11 + 13 + 6 + 1 + (2\alpha^2 + 4\alpha + 3)\sqrt{A}}{2(\alpha + 1)^2(\alpha^2 + \alpha + 1)} \qquad (23),$$

$$A_2 = \frac{-\alpha^6 - 4 - 9 - 16 - 20 - 14 - 5 - (\alpha^3 + 3 + 4 + 3)\sqrt{A}}{2(\alpha + 1)^4} \qquad (24),$$

$$A_3 = \frac{\left\{\begin{array}{c} -\alpha^9 - 5 - 15 - 34 - 58 - 73 - 63 - 31 - 6 + 1 \\ + (-\alpha^6 - 4 - 9 - 14 - 14 - 8 - 1)\sqrt{A} \end{array}\right\}}{2(\alpha + 1)^4(\alpha^2 + \alpha + 1)} \qquad (25),$$

$$C_1 = \frac{\left\{\begin{array}{c} -1 - 5 - 11 - 6 + 14 + 31 + 29 + 17 + 6 + 1 \\ + (-1 - 4 - 5 - 6 - 6 - 4 - 1)\sqrt{A} \end{array}\right\}}{4\alpha^4(\alpha + 1)^2(\alpha^2 + \alpha + 1)} \qquad (26),$$

$$C_2 = \frac{\left\{\begin{array}{c} 1 + 3 + 5 + 6 + 4 + 7 + 11 + 9 + 4 + 1 \\ + (1 + 2 + 1 - 2 - 2 - 2 - 1)\sqrt{A} \end{array}\right\}}{4\alpha^4(\alpha + 1)^2(\alpha^2 + \alpha + 1)} \qquad (27).$$

Expressed in terms of α, the parameters employed by KIEPERT ('Math. Ann.,' 32, p. 128) are given by

$$\tau_{18} + 2 = \frac{1}{\xi_3} = \frac{\alpha^3 + \alpha^2 - 2\alpha - 1 + \sqrt{A}}{2\alpha(\alpha + 1)} \qquad (28),$$

τ_{18} being GIERSTER'S τ ('Math. Ann.,' 14, p. 540),

$$\tau_{18} = \xi_6 = \frac{\alpha^3 - 3\alpha^2 - 6\alpha - 1 + \sqrt{A}}{2\alpha(\alpha + 1)} \qquad (29).$$

$$\xi_3 = \frac{-\alpha^3 - \alpha^2 + 2\alpha + 1 + \sqrt{A}}{4\alpha(\alpha + 1)} \qquad (30),$$

$$\xi_1 = \frac{\alpha^3 + 3\alpha^2 + 0 - 1 - \sqrt{A}}{2\alpha(\alpha + 1)} \qquad (31),$$

$$\xi_2 = \frac{\alpha^3 + 3\alpha^2 + 0 - 1 - \sqrt{A}}{2(\alpha^3 - 3\alpha - 1)} \qquad (32),$$

$$\frac{\xi_1}{\xi_2} = \frac{\alpha^3 - 3\alpha - 1}{\alpha(\alpha + 1)} \qquad (33),$$

$$\xi_4 = \frac{-a^3 - 9a^2 - 6a + 1 + 3\sqrt{A}}{2(a^3 - 3a - 1)} \tag{34},$$

$$\xi_5 = \frac{a^3 - a^2 - 4a - 1 + \sqrt{A}}{2a(a+1)} \tag{35},$$

$$\xi_6 = \frac{a^3 - 3a^2 - 6a - 1 + \sqrt{A}}{2a(a+1)} \tag{36},$$

$$\mathrm{L}\,(2)^{24} = \tag{37}.$$

Then for the section-values

$$\kappa^2 \mathrm{tn}^4 \tfrac{2}{9}\mathrm{K}' = a_1 = \frac{\begin{Bmatrix} q^9 + 1 + 5 + 28 + 66 + 83 + 73 + 29 + 8 + 1 \\ + (q^6 - 2 - 11 - 18 - 14 - 6 - 1)\sqrt{Q} \end{Bmatrix}}{2(q+1)^7(q^2+q+1)} \tag{38},$$

$$\kappa^2 \mathrm{tn}^4 \tfrac{7}{9}\mathrm{K}' = \frac{1}{a_1} \tag{39},$$

$$\kappa^2 \mathrm{tn}^4 \tfrac{4}{9}\mathrm{K}' = a_2 = \frac{q^6 + 4 + 9 + 16 + 14 + 6 + 1 + (q^3 + 3q^2 + 4q + 1)\sqrt{Q}}{2(q+1)(q^2+q+1)} \tag{40},$$

$$\kappa^2 \mathrm{tn}^4 \tfrac{5}{9}\mathrm{K}' = \frac{1}{a_2} = \frac{1 + 4 + 9 + 16 + 14 + 6 + 1 - (1 + 3 + 4 + 1)\sqrt{Q}}{2q^3(q^3 - 3q - 1)} \tag{41}.$$

$$1 - \frac{m}{\alpha} = \left(\frac{q^3 + q^2 + 0 - 1 - \sqrt{Q}}{2q}\right)^2 \tag{42},$$

$$\alpha = \frac{-q^9 - 3 - 5 - 6 - 4 - 7 - 11 - 9 - 4 - 1 + (-q^6 - 2 - 1 + 2 + 2 + 2 + 1)\sqrt{Q}}{16q^4(q+1)^2(q^2+q+1)} \tag{43}.$$

$$\gamma_9 = 0, \quad \lambda = \frac{\gamma_5}{\gamma_4} = \frac{y - x}{y} = z = p(1-p) \tag{44},$$

$$p = p_9 = \frac{q^3 + q^2 + 0 - 1 + \sqrt{Q}}{2q(q+1)^2} \tag{45}.$$

$$\left(\frac{\Theta\frac{2}{9}\mathrm{K}'}{\Theta 0}\right)^9 = c_1^{\,9} = \frac{\alpha^3}{z}\left(1 - \frac{m}{\alpha}\right)^{\frac{3}{2}} \tag{46},$$

$$\left(\frac{\Theta\frac{4}{9}\mathrm{K}'}{\Theta 0}\right)^9 = c_2^{\,9} = \frac{\alpha^3}{z^4}\left(\frac{\alpha}{\alpha - m}\right)^3 \tag{47},$$

$$\frac{\Theta\frac{2}{3}\mathrm{K}'}{\Theta 0} = c_3 = \frac{q^3 + q^2 - 2q - 1 + \sqrt{Q}}{-4q(q+1)}$$

$$= \frac{-1}{\tau_{18} + 2} = \frac{(b^2 - 1)^{\frac{1}{3}}}{2} \tag{48},$$

from the table for $\mu = 6$, § 29.

33. $\mu = 22$, $f = \dfrac{r}{11}$.

The relation $D_{11} = 0$ in (21), § 22, gives a C_8 in (p, ϵ); but putting

$$\epsilon = \frac{1}{x+1}, \quad p = \frac{1}{y+1} \qquad (1),$$

it becomes

$$(x+1)^2 y^5 + (4x^2 + 9x + 4) y^4 - (x+2)(3x^2 - 3x - 2) y^3 \\ + x(x+2)(x^2 - 7x - 2) y^2 + 4x^3 (x+2) y - x^4 (x+2) = 0 \qquad (2),$$

a C_7 in (x, y) with triple point at the origin; so putting $y = -sx$,

$$s^5 x^4 + (2s^3 - 4s^2 - 3s - 1) s^2 x^3 + (s^5 - 9s^4 - 3s^3 + 5s^2 + 4s + 1) x^2 \\ - 2(2s^4 - 4s^3 - 8s^2 - 4s - 1) x + 4s^2 (s+1) = 0 \qquad (3),$$

a quartic for x, which can be resolved

$$[2s^3 x^2 + (2s^3 - 4s^2 - 3s - 1) x - 2(2s^2 + 2s + 1)]^2 = \{(s-1)x - 2\}^2 \mathrm{S} \qquad (4),$$

$$\mathrm{S} = 4s(s+1)^2 + 1 \qquad (5).$$

Writing it

$$[2sx(s^2 x - 2s - 1) + (2s^2 + 2s + 1)\{(s-1)x - 2\}]^2 \\ = \{(s-1)x - 2\}^2 \mathrm{S} \qquad (6),$$

and putting

$$\frac{2sx(s^2 x - 2s - 1)}{(s-1)x - 2} = -2s^2 (t+1) \qquad (7),$$

$$(-2s^2 t + 2s + 1)^2 = \mathrm{S}, \quad t = \frac{2s + 1 + \sqrt{\mathrm{S}}}{2s^2} \qquad (8),$$

so that, with

$$u = \int^\infty \frac{ds}{\sqrt{\mathrm{S}}}, \quad s = s(u) \qquad (9),$$

$$t = s(u - \tfrac{2}{5}\omega) \qquad (10).$$

This is the elliptic integral of which the ikosahedron irrationality η of Klein is unity; it is curious also that this integral occurs as one of Abel's numerical exercises ('Œuvres,' 1, p. 142).

Then to connect up with $\mu = 11$, as in L.M.S., 27, p. 469,

$$1 + c_{11} = p_{11} = \frac{s(2v) - s(v)}{s(5v) - s(v)} = \frac{(x-y)^2}{(y+1)(xy - 2x + 2y)} \qquad (11),$$

after reduction; and

$$\sqrt{\mathrm{C}} = 1 - 2z_{11}, \quad z_{11} = \frac{s(3v) - s(2v)}{s(3v) - s(v)} \qquad (12),$$

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reducing to

$$z_{11} = \frac{-(y+2)(x-y)}{(y+1)(xy-2x+2y)}, \quad \frac{z_{11}}{p_{11}} = \frac{y+2}{y-x} \qquad (13),$$

where x and y are given in terms of s by equation (64).

Substituting these values of c and $\sqrt{C}$ in terms of x and y in the expression for $I(v)$ in (42), § 11, for $\mu = 11$, we obtain a result applicable to $\mu = 22$, and associated functions of the Second Stage.

34. The next two cases of $\mu = 13$, 26 and $\mu = 15$, 30 still present analytical difficulties not yet surmounted, although $\mu = 30$ could be treated by the trisection method of § 6, applied to $\mu = 10$.

35. When μ is of the form $4n$, so that

$$v = \omega_1 + \frac{2r+1}{2n}\omega_3, \quad \text{or} \quad K + \frac{2r+1}{2n}K'i, \quad f = \frac{2r+1}{2n} \qquad (1),$$

then $\frac{1}{2}\mu v = 2nv$ is congruent to ω_3, as in Case III., and $I(v)$ is given by

$$I(v) = \frac{1}{2n}\cos^{-1}\frac{P_1 s^{n-1} + P_2 s^{n-2} + \ldots + P_n}{(\sigma - s)^n}\sqrt{(s - s_3)}$$

$$= \frac{1}{2n}\sin^{-1}\frac{s^{n-1} + Q_2 s^{n-2} + \ldots + Q_n}{(\sigma - s)^n}\sqrt{(s_1 - s \,.\, s_2 - s)}$$

$$= \int_s^{s_2} \frac{-P(v)(\sigma - s) + \frac{1}{2}\sqrt{-\Sigma}}{\sigma - s}\frac{ds}{\sqrt{S}} \qquad (2),$$

$$s_1 > \sigma > s_2 > s > s_3 \qquad (3).$$

Here the degree is halved by putting

$$s - s_3 = x^2 \qquad (4),$$

so that

$$\int \frac{ds}{\sqrt{S}} = \int \frac{dx}{\sqrt{X_1 X_2}} \qquad (5),$$

and

$\frac{1}{4}\mu$ odd	$\frac{1}{4}\mu$ even	
$X_1 = C_0 + C_1 x - x^2$	$X_1 = C_0 + C_1 x + x^2$	
$= (x_1 - x)(x_2 + x)$	$= (x_1 + x)(x_2 + x)$	(6),
$X_2 = C_0 - C_1 x - x^2$	$X_2 = C_0 - C_1 x + x^2$	
$= (x_1 + x)(x_2 - x)$	$= (x_1 - x)(x_2 - x)$	(7),

also

$$x_2 = \kappa x_1 \qquad (8),$$

and putting

$$\delta = \sigma - s_3 = (s_1 - s_3)\,\mathrm{dn}^2 fK' = x_1{}^2\,\mathrm{dn}^2 fK' \qquad (9),$$

and

$$\mathrm{P}(v) = x_1 \mathrm{zn} f \mathrm{K}' \tag{10},$$

$$\mathrm{I}(v) = \int_x^{x_2} \frac{-\mathrm{P}(v)(\delta - x^2) + \frac{1}{2}\sqrt{-\Sigma}}{\delta - x^2} \frac{dx}{\sqrt{\mathrm{X}_1\mathrm{X}_2}}$$

$$= \frac{1}{n}\cos^{-1}\frac{\mathrm{R}_0 + \mathrm{R}_1 x + \ldots + \mathrm{R}_{n-1}x^{n-1}}{(\delta - x^2)^{\frac{1}{2}n}}\sqrt{\tfrac{1}{2}\mathrm{X}_1}$$

$$= \frac{1}{n}\sin^{-1}\frac{\mathrm{R}_0 - \mathrm{R}_1 x + \ldots + (-1)^{n-1}\mathrm{R}_{n-1}x^{n-1}}{(\delta - x^2)^{\frac{1}{2}n}}\sqrt{\tfrac{1}{2}\mathrm{X}_2} \tag{11}.$$

Thus for the vector of the corresponding herpolhode

$$\mathrm{M}\frac{\rho}{k} = \sqrt{(\delta - x^2)}, \quad x = x_2 \mathrm{sn}(\mathrm{K} - mt) \tag{12},$$

$$\mathrm{M}\frac{\rho}{k} e^{(pt-\varpi)i} = \{(\mathrm{R}_0 + \mathrm{R}_1 x + \ldots)\sqrt{\tfrac{1}{2}\mathrm{X}_1} + i(\mathrm{R}_0 - \mathrm{R}_1 x + \ldots)\sqrt{\tfrac{1}{2}\mathrm{X}_2}\}^{\frac{1}{n}} \tag{13},$$

and in the associated hodograph described by the axis of a top

$$\tfrac{1}{2}\mathrm{M}^2 \sin\theta e^{(pt-\psi)i}$$
$$= \{(\mathrm{A}_0 + \mathrm{A}_1 x + \ldots \mathrm{A}_{2n-1}x^{2n-1})\sqrt{\tfrac{1}{2}\mathrm{X}_1} + i(\mathrm{A}_0 - \mathrm{A}_1 x + \ldots - \mathrm{A}_{2n-1}x^{2n-1})\sqrt{\tfrac{1}{2}\mathrm{X}_2}\}^{\frac{1}{n}} \tag{14}.$$

36. With $\mu = 4n$, the resolution of T into factors for the spherical catenary must be such that T_1 and T_2 are quadratics in t^2; and thus

$$\mathrm{T}_1 = -\left\{k - 1 \pm 2\sqrt{\frac{k-h}{k+h}}\,t + (k+1)t^2\right\} \tag{15},$$

$$\mathrm{T}_2 = \left\{k + 1 \pm 2\sqrt{\frac{k+h}{k-h}}\,t + (k-1)t^2\right\} \tag{16},$$

$$\mathrm{T}_1 = -(k-1)^2 + 2\left(2\frac{k-h}{k+h} - k^2 + 1\right)t^2 - (k+1)^2 t^4 \tag{17},$$

$$\mathrm{T}_2 = (k+1)^2 - 2\left(2\frac{k+h}{k-h} - k^2 + 1\right)t^2 + (k-1)^2 t^4 \tag{18}.$$

37. The simplest cases may be cited before proceeding to the general theory.

$$\mu = 4, \quad v = \mathrm{K} + \tfrac{1}{2}\mathrm{K}'i \tag{1},$$

$$\mathrm{I}(v) = \int_x^\kappa \frac{\frac{1}{2}(1-\kappa)(\kappa + x^2)}{\kappa - x^2} \frac{dx}{\sqrt{(1-x^2)(\kappa^2 - x^2)}}$$

$$= \cos^{-1}\sqrt{\frac{(1-x)(\kappa+x)}{2(\kappa - x^2)}} = \sin^{-1}\sqrt{\frac{(1+x)(\kappa - x)}{2(\kappa - x^2)}} \tag{2};$$

and in the associated dynamical applications

$$\mathrm{M}\frac{\rho}{k} = \sqrt{(\kappa - x^2)}, \quad x = \kappa \operatorname{sn}(\mathrm{K} - mt) \tag{3},$$

$$\mathrm{M}\frac{\rho}{k} e^{(pt-\varpi)i} = \sqrt{\tfrac{1}{2}(1-x)(\kappa+x)} + i\sqrt{\tfrac{1}{2}(1+x)(\kappa-x)} \tag{4},$$

reducing to HALPHEN's algebraical herpolhode when the secular term pt is cancelled by putting $p = 0$.

In the associated motion of the top ('Annals of Mathematics,' vol. 5)

$$\tfrac{1}{2}\mathrm{M}^2 \sin\theta e^{(pt-\psi)i} = \left(\frac{p}{m} - \frac{1-\kappa}{2} - x\right)\sqrt{\tfrac{1}{2}(1-x)(\kappa+x)}$$

$$+ i\left(\frac{p}{m} - \frac{1-\kappa}{2} + x\right)\sqrt{\tfrac{1}{2}(1+x)(\kappa - x)} \tag{5}.$$

But the quadric transformation

$$z = \frac{(1-\kappa)x}{\kappa - x^2} = \operatorname{cn}[(1+\kappa)mt, \gamma], \quad \gamma = \frac{2\sqrt{\kappa}}{1+\kappa} \tag{6},$$

makes

$$\mathrm{I}(v) = \tfrac{1}{2}\int_z^1 \frac{dz}{\sqrt{(1-z^2)}} = \cos^{-1}\sqrt{\frac{1+z}{2}} = \sin^{-1}\sqrt{\frac{1-z}{2}} \tag{7},$$

thus effecting the identification with the result, p. 147, 'Œuvres,' 2, when the sign of ABEL's R is changed so as to obtain the circular form.

For the division-values

$$\operatorname{sn}\tfrac{1}{2}\mathrm{K}' = \sqrt{\frac{1}{1+\kappa}}, \quad \operatorname{cn}\tfrac{1}{2}\mathrm{K}' = \sqrt{\frac{\kappa}{1+\kappa}}, \quad \operatorname{dn}\tfrac{1}{2}\mathrm{K}' = \sqrt{\kappa} \tag{8};$$

$$\operatorname{zn}\tfrac{1}{2}\mathrm{K}' = \operatorname{zd}\tfrac{1}{2}\mathrm{K}' = \tfrac{1}{2}(1-\kappa), \quad \operatorname{zs}\tfrac{1}{2}\mathrm{K}' = -\operatorname{zc}\tfrac{1}{2}\mathrm{K}' = \tfrac{1}{2}(1+\kappa) \tag{9};$$

$$\frac{\Theta\frac{1}{2}\mathrm{K}'}{\Theta 0} = \frac{(1+\kappa)^{\frac{1}{4}}}{2^{\frac{1}{4}}\kappa^{\frac{3}{8}}}, \quad \frac{\mathrm{H}\frac{1}{2}\mathrm{K}'}{\Theta 0} = \frac{(1-\kappa)^{\frac{1}{4}}}{2^{\frac{1}{4}}\kappa^{\frac{3}{8}}}, \quad \frac{\mathrm{H}\frac{1}{2}\mathrm{K}'}{\mathrm{H}\mathrm{K}'} = \frac{\kappa^{\frac{1}{4}}}{2^{\frac{1}{4}}(1+\kappa)^{\frac{1}{4}}} \tag{10}.$$

38. $$\mu = 8, \quad v = \mathrm{K} + \tfrac{1,3}{4}\mathrm{K}'i,$$

$$\mathrm{I}(v) = \int_x^\kappa \frac{-\mathrm{P}(v)(\delta - x^2) + \mathrm{Q}(v)}{\delta - x^2} \frac{dx}{\sqrt{\mathrm{X}_1 \mathrm{X}_2}}$$

$$= \frac{1}{2}\cos^{-1}\frac{\frac{1}{2}(1+a)^2 - x}{\delta - x^2}\sqrt{\tfrac{1}{2}(1+x)(\kappa + x)}$$

$$= \frac{1}{2}\sin^{-1}\frac{\frac{1}{2}(1+a)^2 + x}{\delta - x^2}\sqrt{\tfrac{1}{2}(1-x)(\kappa - x)} \qquad (1),$$

where the octahedron irrationality $o = \sqrt{\kappa}$ is given in terms of a by

$$o = \tfrac{1}{2}\left(\frac{1}{a} - a\right) \qquad (2),$$

unchanged by the substitution $\left(a, -\dfrac{1}{a}\right)$, which interchanges $f = \frac{1}{4}$ and $\frac{3}{4}$.

Also, in the region $\sqrt{2} > a > \sqrt{2} - 1$, $f = \frac{1}{4}$,

$$\delta = \frac{(1+a)^3(1-a)}{4a} = \mathrm{dn}^2 \tfrac{1}{4}\mathrm{K}' \qquad (3),$$

$$\mathrm{P}(v) = \frac{(1 + 2a + 3a^2)(-1 + 2a + a^2)}{16a^2} = \mathrm{zn}\tfrac{1}{4}\mathrm{K}' \qquad (4)$$

$$\mathrm{Q}(v) = \tfrac{1}{2}\sqrt{-\Sigma} = \frac{(1+a)^2(1-a^4)(-1 + 2a + a^2)}{16a^3}$$

$$= \kappa'^2 \mathrm{sn}\tfrac{1}{4}\mathrm{K}' \mathrm{cn}\tfrac{1}{4}\mathrm{K}' \mathrm{dn}\tfrac{1}{4}\mathrm{K}' \qquad (5).$$

The result is obtained by putting

$$s(\omega) - s(4v) = 0, \quad \alpha = \frac{m(1-m)}{1-2m}, \quad \beta = 1 \qquad (6);$$

and the values of s_1, s_2, s_3 are rationalised by putting

$$m = \frac{1+a}{1+2a-a^2}, \quad 1 - 8(1-2m)\alpha = \left(\frac{-1+2a+a^2}{1+2a-a^2}\right)^2 \qquad (7).$$

To normalise the results, a homogeneity factor M is introduced, such that

$$s - s_3 = \mathrm{M}^2 x, \quad \mathrm{M} = \sqrt{(s_1 - s_3)} = \frac{2a^2}{(1+a)(1+2a-a^2)} \qquad (8).$$

The division-values are shown in tabular form, and the numerical test of $a = \frac{1}{2}(\sqrt{5} - 1)$ is required for $\mu = 15$.

Region.	A. $\infty > a > \sqrt{2}+1.$	B. $\sqrt{2}+1 > a > 1.$	C. $1 > a > \sqrt{2}-1.$
$\frac{1}{4}\left(\frac{1}{a} - a\right)^2$	$\frac{1}{\kappa'}$	κ'	κ'
$\frac{(1+a^2)^2(-1+2a+a^2)(1+2a-a^2)}{16a^4}$	$-\frac{\kappa^2}{\kappa'^2}$	κ^2	κ^2
$\frac{(1+a)^3(1-a)}{4a}$	$-\mathrm{sc}^2 \frac{3}{4}\mathrm{K}$	$-\mathrm{cs}^2 \frac{1}{4}\mathrm{K}$	$\mathrm{dn}^2 \frac{1}{4}\mathrm{K}$
$\frac{(1-a)^3(1+a)}{4a^3}$	$-\mathrm{sc}^2 \frac{1}{4}\mathrm{K}$	$-\mathrm{cs}^2 \frac{3}{4}\mathrm{K}$	$\mathrm{dn}^2 \frac{3}{4}\mathrm{K}$
$\frac{1}{2}\left(\frac{1}{a} - a\right)$	$-\mathrm{nd}\,\frac{1}{2}\mathrm{K}$	$-\mathrm{dn}\,\frac{1}{2}\mathrm{K}$	$\mathrm{dn}\,\frac{1}{2}\mathrm{K}$
$\frac{(1+2a+3a^2)(-1+2a+a^2)}{16a^2}$	$\frac{1}{\kappa'}\mathrm{zs}\,\frac{1}{4}\mathrm{K}$	$\mathrm{zs}\,\frac{1}{4}\mathrm{K}$	$\mathrm{zn}\,\frac{1}{4}\mathrm{K}$
$\frac{(3-2a+a^2)(-1+2a+a^2)}{16a^2}$	$\frac{1}{\kappa'}\mathrm{zs}\,\frac{3}{4}\mathrm{K}$	$\mathrm{zs}\,\frac{3}{4}\mathrm{K}$	$\mathrm{zn}\,\frac{3}{4}\mathrm{K}$
$\frac{1}{8}\left(\frac{1}{a} + a\right)^2$	$\frac{1}{\kappa'}\mathrm{zn}\,\frac{1}{2}\mathrm{K}$	$\mathrm{zn}\,\frac{1}{2}\mathrm{K}$	$\mathrm{zn}\,\frac{1}{2}\mathrm{K}$
$\frac{a^{\frac{1}{16}}(1+a)^{\frac{1}{16}}(1-a)^{\frac{9}{16}}}{(1+a^2)^{\frac{3}{8}}(-1+2a+a^2)^{\frac{1}{4}}}$	$\frac{\mathrm{H}\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\mathrm{H}\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$
$\frac{a^{\frac{9}{16}}(1+a)^{\frac{9}{16}}(1-a)^{\frac{1}{16}}}{(1+a^2)^{\frac{3}{8}}(-1+2a+a^2)^{\frac{1}{4}}}$	$\frac{\mathrm{H}\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\mathrm{H}\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$
$\frac{\left(\frac{1}{a} + a\right)^{\frac{1}{2}}}{\left(\frac{1}{a} - a\right)^{\frac{3}{8}}}$	$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta\mathrm{K}}$	$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta 0}$	$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta 0}$
Test by a	$\sqrt{5}+2$	$\frac{1}{2}(\sqrt{5}+1)$	$\frac{1}{2}(\sqrt{5}-1)$

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D. $\sqrt{2}-1>a>0$.	E. $0>a>-\sqrt{2}+1$.	F. $-\sqrt{2}+1>a>-1$.	G. $-1>a>-\sqrt{2}-1$.	H. $-\sqrt{2}-1>a>-\infty$.
$\frac{1}{\kappa'}$	$\frac{1}{\kappa'}$	κ'	κ'	$\frac{1}{\kappa'}$
$-\frac{\kappa^2}{\kappa'^2}$	$-\frac{\kappa^2}{\kappa'^2}$	κ^2	κ^2	$-\frac{\kappa^2}{\kappa'^2}$
$\mathrm{nd}^2\,\frac{1}{4}\mathrm{K}$	$-\mathrm{sc}^2\,\frac{3}{4}\mathrm{K}$	$-\mathrm{cs}^2\,\frac{3}{4}\mathrm{K}$	$\mathrm{dn}^2\,\frac{3}{4}\mathrm{K}$	$\mathrm{nd}^2\,\frac{3}{4}\mathrm{K}$
$\mathrm{nd}^2\,\frac{3}{4}\mathrm{K}$	$-\mathrm{sc}^2\,\frac{1}{4}\mathrm{K}$	$-\mathrm{cs}^2\,\frac{1}{4}\mathrm{K}$	$\mathrm{dn}^2\,\frac{1}{4}\mathrm{K}$	$\mathrm{nd}^2\,\frac{1}{4}\mathrm{K}$
$\mathrm{nd}\,\frac{1}{2}\mathrm{K}$	$-\mathrm{nd}\,\frac{1}{2}\mathrm{K}$	$-\mathrm{dn}\,\frac{1}{2}\mathrm{K}$	$\mathrm{dn}\,\frac{1}{2}\mathrm{K}$	$\mathrm{nd}\,\frac{1}{2}\mathrm{K}$
$-\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{3}{4}\mathrm{K}$	$-\frac{1}{\kappa'}\,\mathrm{zs}\,\frac{3}{4}\mathrm{K}$	$-\mathrm{zs}\,\frac{3}{4}\mathrm{K}$	$-\mathrm{zn}\,\frac{3}{4}\mathrm{K}$	$\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{1}{4}\mathrm{K}$
$-\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{1}{4}\mathrm{K}$	$-\frac{1}{\kappa'}\,\mathrm{zs}\,\frac{1}{4}\mathrm{K}$	$-\mathrm{zs}\,\frac{1}{4}\mathrm{K}$	$-\mathrm{zn}\,\frac{1}{4}\mathrm{K}$	$\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{3}{4}\mathrm{K}$
$\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{1}{2}\mathrm{K}$	$\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{1}{2}\mathrm{K}$	$\mathrm{zn}\,\frac{1}{2}\mathrm{K}$	$\mathrm{zn}\,\frac{1}{2}\mathrm{K}$	$\frac{1}{\kappa'}\,\mathrm{zn}\,\frac{1}{2}\mathrm{K}$
$\frac{\Theta\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\mathrm{H}\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\mathrm{H}\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$
$\frac{\Theta\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\mathrm{H}\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\mathrm{H}\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{1}{4}\mathrm{K}}{\mathrm{HK}}$	$\frac{\Theta\,\frac{3}{4}\mathrm{K}}{\mathrm{HK}}$
$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta\mathrm{K}}$	$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta\mathrm{K}}$	$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta\mathrm{K}}$	$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta 0}$	$\frac{\Theta\,\frac{1}{2}\mathrm{K}}{\Theta\mathrm{K}}$
$\sqrt{5}-2$	$-\sqrt{5}+2$	$-\frac{1}{2}(\sqrt{5}-1)$	$-\frac{1}{2}(\sqrt{5}+1)$	$-\sqrt{5}-2$

39. ABEL's form in 'Œuvres,' 1, p. 142, 2, p. 160, for $n = 2$ becomes the same as in 2, p. 148, by replacing his $x + \frac{1}{4}a$ by x, his a by $2(q + q')$, and b by $-\frac{1}{4}(q - q')^2$.

But now in our notation for $\mu = 8$, equation (14) § 25 becomes

$$Z_2 = [z + m(1 - m)]^2 - (1 - 2m)^2 z \qquad (1);$$

and putting

$$m = \frac{1 + a}{1 + 2a - a^2}, \quad 1 - m = \frac{a(1 - a)}{1 + 2a - a^2}, \quad 1 - 8m + 8m^2 = \left(\frac{-1 + 2a + a^2}{1 + 2a - a^2}\right)^2 \qquad (2)$$

and replacing z by $\frac{z}{M}$, where $M = 1 + 2a - a^2$, the roots of $Z = Z_1 Z_2 = 0$ are

$$z_0 = (1 + a)^2, \quad z_1 = a^2(1 + a)^2, \quad z_2 = (1 - a)^2, \quad z_3 = a^2(1 - a)^2 \qquad (3).$$

A quadric transformation is required in ABEL's result to obtain our I (v), namely, with the δ above, and the sign of Z_1 changed to obtain the circular form,

$$z_0 - z = \frac{(1 + a)^3(1 - a)(1 - x^2)}{\delta - x^2} \qquad (4),$$

$$z_1 - z = \frac{(1 + a^2)(-1 + 2a + a^2)\delta}{\delta - x^2} \qquad (5),$$

$$z_2 - z = \frac{(1 + a^2)(-1 + 2a + a^2)x^2}{\delta - x^2} \qquad (6),$$

$$z - z_3 = \frac{4a^3(1 - a^2)^4(\kappa^2 - x^2)}{\delta - x^2} \qquad (7),$$

and

$$Z_1 = (z_0 - z)(z - z_3) = \frac{4a^3(1 + a)^7(1 - a)^5(1 - x^2)(\kappa^2 - x^2)}{(\delta - x^2)^2} \qquad (8),$$

$$Z_2 = (z_1 - z)(z_2 - z) = \frac{(1 + a^2)^2(-1 + 2a + a^2)^2\delta x^2}{(\delta - x^2)^2} \qquad (9),$$

so that $\sqrt{Z_2}$ is rational.

Afterwards the degree is halved, as ABEL's integral is 2I (v).

A similar quadric transformation and halving of degree will be required for all even values of ABEL's n, to reduce his results to our form, involving heavy algebraical work.

ABEL's integral ('Œuvres,' 2, p. 148), changed into the circular form, can be utilised for the construction of an algebraical orbit described under the central attraction

$$P = \frac{\mu}{r^2} + \mu_0 + \mu_1 r \qquad (10),$$

in the form

$$N \cos 2\theta = (u + q + 2q') \sqrt{(u^2 + 2qu - q^2 - 2qq')} \tag{11}$$

$$N \sin 2\theta = (u + 2q + q') \sqrt{(- u^2 - 2q'u + 2qq' + q'^2)} \tag{12}$$

$$N^2 = (q' - q)^3 (q' + q) \tag{13}$$

so that the condition

$$\theta = \int \frac{u\, du}{\sqrt{U}} \tag{14}$$

$$U = (u^2 + 2qu - q^2 - 2qq') (- u^2 - 2q'u + 2qq' + q'^2) \tag{15}$$

is satisfied by taking

$$\frac{\mu}{h^2} = - q - q', \quad \frac{H}{h^2} = q^2 + q'^2, \quad \frac{\mu_0}{h^2} = - 3qq' (q + q')$$

$$\frac{\mu_1}{h^2} = qq' (q + 2q') (2q + q') \tag{16}$$

A degenerate circular orbit $u = q$ is obtained by taking

$$q + q' = 0, \quad \mu = 0, \quad \mu_0 = 0, \quad H = 2q^2h^2,$$

$$P = h^2q^3, \quad v^2 = h^2q^2 \tag{17}$$

But the integral in 'Œuvres,' 2, p. 147, putting $x + q = u$, requires $\mu = 0$, $\mu_0 = 0$, so that we merely obtain a conic for $P = \mu_1 r$.

40. $$\mu = 12, \quad v = K + \frac{1 \text{ or } 5}{6} K'i, \quad f = \tfrac{1}{6} \text{ or } \tfrac{5}{6}.$$

$$I(v) = \int_x^{x_2} \frac{- P(v)(\delta - x^2) + Q(v)}{a + a^2 + a^3 - x^2} \frac{dx}{\sqrt{X_1 X_2}}$$

$$= \tfrac{1}{3} \cos^{-1} \frac{1 + a + a^2 + (1 - a) x - x^2}{(a + a^2 + a^3 - x^2)^{\frac{3}{2}}} \sqrt{\tfrac{1}{2} X_1}$$

$$= \tfrac{1}{3} \sin^{-1} \frac{1 + a + a^2 - (1 - a) x - x^2}{(a + a^2 + a^3 - x^2)^{\frac{3}{2}}} \sqrt{\tfrac{1}{2} X_2} \tag{1}$$

$$X_1, X_2 = a^3 (1 + a + a^2) \pm \tfrac{1}{2} (1 + a)^3 (1 - a) x - x^2 \tag{2}$$

$$P(v) = \tfrac{1}{12} (1 - a) (5 + 3a + 3a^2 + a^3) = x_1 \text{zn} f K' \tag{3}$$

$$Q(v) = \tfrac{1}{2} (1 - a^4) (a + a^2 + a^3) = x_1^3 \kappa'^2 \text{sn} f K' \text{cn} f K' \text{dn} f K' \tag{4}$$

results obtained by putting

$$s(\omega) - s(6v) = 0, \quad \alpha = \frac{m(1-m)(1-3m+3m^2)}{(1-2m)(1-2m+2m^2)}, \quad \beta = \frac{1-3m+3m^2}{1-2m+2m^2} \quad (5),$$

$$\mathrm{M} = m = \frac{1}{1-\alpha} \quad (6).$$

Treated by the trisection of $\mu = 4$, putting $y = 0$ in equation (8), § 14,

$$3t^4 + (1-8x)t^3 + 6x^2t^2 - x^4 = 0 \quad (7),$$

and putting

$$t = \frac{x}{b},$$

$$x = \frac{b}{(b-1)^3(b+3)}, \quad t = \frac{1}{(b-1)^3(b+3)} \quad (8),$$

$$\tau_4 = \frac{1}{8x} = \frac{(b-1)^3(b+3)}{8b} \quad (9),$$

and

$$\tau_4 = \frac{(\tau-4)^3\tau}{8(\tau-3)}, \quad \tau = \tau_{12} \quad (10),$$

so that

$$b = \tau_{12} - 3 \quad (11),$$

and b is the equivalent of y in Klein-Fricke, 'Modulfunctionen,' 1, p. 688, while

$$x = -\tau_{18} - 2 \quad (12),$$

$$2\tau_6 + 8 = (\tau_{18}+2)^3 = y^2 = (\tau_{12}-3)^2 \quad (13),$$

$$x^3 + y^2 = 1 \quad (14),$$

and Gierster's

$$\tau_{12} = \frac{36}{y^2-9} = \frac{36}{-x^3-8} \quad (15).$$

Also

$$2\tau_4 = (\tfrac{1}{o} - o)^2 = \frac{(1+\alpha)^6(1-\alpha)^2}{4\alpha^3(1+\alpha+\alpha^2)} \quad (16),$$

so that

$$b = \frac{1}{\alpha} + 1 + \alpha \quad (17).$$

In Kiepert's notation, 'Math. Ann.,' 32, p. 104,

$$\xi_1 = 1 - \frac{4}{b-1}, \text{ \&c.} \quad (18).$$

Putting

$$\mathrm{A} = (1+\alpha^2)(1+4\alpha+\alpha^2) \quad (19),$$

the section values are shown in the table :—

	$0 < a < 1.$	$1 < a < \infty.$
$\dfrac{(1+a)^3(1-a)}{2a\sqrt{(a+a^2+a^3)}}$	$\frac{1}{o} - o$	$-\frac{1}{o} + o$
$\dfrac{(1+a^2)\sqrt{A}}{2a\sqrt{(a+a^2+a^3)}}$	$\frac{1}{o} + o$	$\frac{1}{o} + o$
$\dfrac{-(1+a)^3(1-a)+(1+a^2)\sqrt{A}}{(1+a)^3(1-a)+(1+a^2)\sqrt{A}}$	κ	$\dfrac{1}{\kappa}$
a	$\dfrac{\mathrm{dn}\,\frac{5}{6}\mathrm{K}'}{o}$	$\dfrac{\mathrm{dn}\,\frac{1}{6}\mathrm{K}'}{o}$
$\dfrac{(1+a)\sqrt{1+4a+a^2}-(1-a)\sqrt{1+a^2}}{(1+a)\sqrt{1+4a+a^2}+(1-a)\sqrt{1+a^2}}$	$\mathrm{dn}\,\frac{1}{3}\mathrm{K}'$	$\mathrm{nd}\,\frac{1}{3}\mathrm{K}'$
$\dfrac{(1+a)\sqrt{1+4a+a^2}-(1-a)\sqrt{1+a^2}}{2(1+a)\sqrt{1+a^2}}$	$\mathrm{cn}\,\frac{1}{3}\mathrm{K}'$	$\mathrm{sn}\,\frac{2}{3}\mathrm{K}'$
$\dfrac{(1+a)\sqrt{1+4a+a^2}+(1-a)\sqrt{1+a^2}}{2(1+a)\sqrt{1+a^2}}$	$\mathrm{sn}\,\frac{2}{3}\mathrm{K}'$	$\mathrm{cn}\,\frac{1}{3}\mathrm{K}'$
$\dfrac{(1+a)\sqrt{1+a^2}-(1-a)\sqrt{1+4a+a^2}}{(1+a)\sqrt{1+a^2}+(1-a)\sqrt{1+4a+a^2}}$	$\mathrm{dn}\,\frac{2}{3}\mathrm{K}'$	$\mathrm{nd}\,\frac{2}{3}\mathrm{K}'$
$\dfrac{(1+a)\sqrt{1+a^2}-(1-a)\sqrt{1+4a+a^2}}{2(1+a)\sqrt{1+a^2}}$	$\mathrm{cn}\,\frac{2}{3}\mathrm{K}'$	$\mathrm{sn}\,\frac{1}{3}\mathrm{K}'$
$\dfrac{(1+a)\sqrt{1+a^2}+(1-a)\sqrt{1+4a+a^2}}{2(1+a)\sqrt{1+a^2}}$	$\mathrm{sn}\,\frac{1}{3}\mathrm{K}'$	$\mathrm{cn}\,\frac{2}{3}\mathrm{K}'$
$\dfrac{-(1+a)^3(1-a)+(1+a^2)\sqrt{A}}{4a\sqrt{(a+a^2+a^3)}}$	$\mathrm{dn}\,\frac{1}{2}\mathrm{K}'$	$\mathrm{nd}\,\frac{1}{2}\mathrm{K}'$
$\dfrac{(1-a)(5+3a+3a^2+a^3)}{12a\sqrt{(a+a^2+a^3)}}$	$\dfrac{\mathrm{zn}\,\frac{1}{6}\mathrm{K}'}{o}$	$\dfrac{\mathrm{zn}\,\frac{5}{6}\mathrm{K}'}{o}$
$\dfrac{(1-a)(1+3a+3a^2+5a^3)}{12a\sqrt{(a+a^2+a^3)}}$	$\dfrac{\mathrm{zn}\,\frac{5}{6}\mathrm{K}'}{o}$	$\dfrac{\mathrm{zn}\,\frac{1}{6}\mathrm{K}'}{o}$
$\dfrac{(1+a+a^2)^2}{3a\sqrt{(a+a^2+a^3)}}$	$\dfrac{\mathrm{zs}\,\frac{1}{3}\mathrm{K}'}{o}$	$\dfrac{\mathrm{zs}\,\frac{1}{3}\mathrm{K}'}{o}$
$\dfrac{(1+a)^3(1-a)}{4a\sqrt{(a+a^2+a^3)}}$	$\dfrac{\mathrm{zn}\,\frac{1}{2}\mathrm{K}'}{o}$	$-\dfrac{\mathrm{zn}\,\frac{1}{2}\mathrm{K}'}{o}$
$\dfrac{a^{22}(a+a^2+a^3)^3}{(1-a)^6(1+a)^{10}(1+a^2)^8}$	$\left(\dfrac{\Theta\,\frac{1}{6}\mathrm{K}'}{\mathrm{HK}'}\right)^{24}$	$\left(\dfrac{\Theta\,\frac{5}{6}\mathrm{K}'}{\mathrm{HK}'}\right)^{24}$
$\dfrac{(a+a^2+a^3)^3}{a^2(1-a)^6(1+a)^{10}(1+a^2)^8}$	$\left(\dfrac{\Theta\,\frac{5}{6}\mathrm{K}'}{\mathrm{HK}'}\right)^{24}$	$\left(\dfrac{\Theta\,\frac{1}{6}\mathrm{K}'}{\mathrm{HK}'}\right)^{24}$

41. In the general case a distinction must be made between $\frac{1}{4}\mu$ odd or even.

I. $$\tfrac{1}{4}\mu = 2n + 1, \quad \mu = 8n + 4.$$

$$v = \mathrm{K} + \frac{2r+1}{4n+2}\mathrm{K}'i, \quad \text{or} \quad v = \mathrm{K} + f\mathrm{K}'i, \quad f = \frac{2r+1}{4n+2} \qquad (1);$$

$$\mathrm{X}_1 = \mathrm{C}_0 + 2\mathrm{C}_1 x - x^2 = (x_1 - x)(x_2 + x) \qquad (2),$$

$$\mathrm{X}_2 = \mathrm{C}_0 - 2\mathrm{C}_1 x - x^2 = (x_1 + x)(x_2 - x) \qquad (3),$$

$$x_2 = \kappa x_1 \qquad (4),$$

$$\mathrm{C}_0 = \sqrt{(s_1 - s_3)(s_2 - s_3)} = s(2n+1)v - s(\omega_3) \qquad (5),$$

$$2\mathrm{C}_1 = \sqrt{(s_1 - s_3)} - \sqrt{(s_2 - s_3)} = 2\mathrm{P}(2n+1)v \qquad (6),$$

$$2\mathrm{C}_0\mathrm{C}_1 = \tfrac{1}{2}i\wp'(2n+1)v = \mathrm{Q}(2n+1)v \qquad (7),$$

$$\left(\frac{1}{o} - o\right)^2 = \frac{\mathrm{C}_1{}^2}{\mathrm{C}_0} = \tfrac{1}{4}\mathrm{L}(4)^8 \qquad (8),$$

where o denotes the octahedron irrationality $\sqrt{\kappa}$; so that if the expressions are normalised by the homogeneity factor $\mathrm{M} = \sqrt{\mathrm{C}_0}$,

$$\mathrm{X}_1, \mathrm{X}_2 = 1 \pm \mathrm{C}x - x^2, \quad \mathrm{C} = \frac{1}{o} - o \qquad (9),$$

$$x_1 = \frac{1}{o}, \quad x_2 = o, \quad x = o\,\mathrm{sn}(\mathrm{K} - mt) \qquad (10);$$

$$\mathrm{dn}^2 f\mathrm{K}' = \frac{\sigma - s_3}{s_1 - s_3} = \frac{\mathrm{C}_0\mathrm{D}}{s_1 - s_3} = \kappa\mathrm{D} = o^2\mathrm{D}, \quad \mathrm{zn} f\mathrm{K}' = \frac{\mathrm{P}(v)}{\sqrt{(s_1 - s_3)}} = \frac{o\mathrm{P}(v)}{\mathrm{M}} \qquad (11).$$

Now

$$\mathrm{I}(v) = \int_x^{x_2} \frac{-\dfrac{\mathrm{P}(v)}{\mathrm{M}}(\delta - x^2) + \dfrac{\mathrm{Q}(v)}{\mathrm{M}^3}}{\mathrm{D} - x^2}\frac{dx}{\sqrt{\mathrm{X}_1\mathrm{X}_2}}$$

$$= \frac{1}{2n+1}\cos^{-1}\frac{\mathrm{R}_0 + \mathrm{R}_1 x + \ldots + \mathrm{R}_{2n}x^{2n}}{(\mathrm{D} - x^2)^{n+\frac{1}{2}}}\sqrt{\tfrac{1}{2}\mathrm{X}_1}$$

$$= \frac{1}{2n+1}\sin^{-1}\frac{\mathrm{R}_0 + \mathrm{R}_1 x + \ldots + \mathrm{R}_{2n}x^{2n}}{(\mathrm{D} - x^2)^{n+\frac{1}{2}}}\sqrt{\tfrac{1}{2}\mathrm{X}_2} \qquad (12).$$

$$\mathrm{R}_0{}^2\mathrm{C}_0 = \mathrm{D}^{2n+1},$$

$$\mathrm{R}_{2n-1} = -\{(2n+1)\mathrm{P}(v) - \mathrm{P}(2n+1)v\}\mathrm{R}_{2n}, \quad \mathrm{R}_{2n} = (-1)^n,$$

$$\frac{\mathrm{Q}(v)}{\mathrm{M}^3} = \frac{\kappa'^2 \mathrm{sn} f\mathrm{K}' \mathrm{cn} f\mathrm{K}' \mathrm{dn} f\mathrm{K}'}{\kappa^{\frac{3}{2}}} \qquad (13).$$

Next putting

$$\frac{\alpha - m}{\alpha - m + 1} = c^4 \quad (14),$$

$$C_0^2 = (s_1 - s_3)(s_2 - s_3) = \frac{m^4\alpha^2(\alpha - m + 1)}{(\alpha - m)^3} = \frac{m^4\alpha^2}{(\alpha - m)^2}\frac{1}{c^4} \quad (15),$$

and with

$$\delta = s(v) - s(\omega_3) = -\frac{m^2\alpha}{\alpha - m} \quad (16),$$

$$\delta \text{ positive}, \quad v = \omega_1 + f\omega_3, \qquad \delta \text{ negative}, \quad v = f\omega_3.$$

$$C_0 = -\frac{m^2\alpha}{\alpha - m}\frac{1}{c^2} = \frac{\delta}{c^2} \quad (17);$$

so that when normalised by $M = \sqrt{C_0}$, D becomes c^2 and R_0 becomes c^{2n+1}.

In the sequel it is convenient to put

$$c = \frac{a - b}{a + b} = \frac{\operatorname{dn} f K'}{o} = \frac{o}{\operatorname{dn}(1 - f) K'} \quad (18),$$

$$p = 1 - 2m = \frac{a - b^2}{b(1 - a)} \quad (19),$$

$$p - 1 = \frac{(a - b)(1 + b)}{b(1 - a)}, \quad p + 1 = \frac{(a + b)(1 - b)}{b(1 - a)} \quad (20);$$

so that

$$\alpha - m = \frac{c^4}{1 - c^4} = \frac{(a - b)^4}{8ab(a^2 + b^2)} \quad (21),$$

$$\begin{aligned} \alpha &= \frac{1 - p}{2} + \frac{c^4}{1 - c^4} = \frac{1 + c^4}{2(1 - c^4)} - \frac{p}{2} \\ &= \frac{a^4 + 6a^2b^2 + b^4}{8ab(a^2 + b^2)} - \frac{a - b^2}{2b(1 - a)} \\ &= -\frac{(a^2 - b^2)\{3a^2 + a^3 + (1 + 3a)b^2\}}{8ab(1 - a)(a^2 + b^2)} \end{aligned} \quad (22).$$

$$\begin{aligned} \beta &= \frac{(1 - 2m)\alpha}{m(1 - m)} = -\frac{4p\alpha}{p^2 - 1} \\ &= \frac{(a - b^2)\{3a^2 + a^3 + (1 + 3a)b^2\}}{2a(1 - b^2)(a^2 + b^2)} \end{aligned} \quad (23),$$

$$\epsilon = 2\beta - 1 = \frac{2a^3 + a^4 - (1 + 2a)b^4}{a(1 - b^2)(a^2 + b^2)} \quad (24),$$

$$\epsilon - 1 = \frac{(1 + a)(a^2 - b^2)(a + b^2)}{a(1 - b^2)(a^2 + b^2)} \quad (25),$$

$$\epsilon + 1 = \frac{\{3a^2 + a^3 + (1 + 3a) b^2\} (a - b^2)}{a (1 - b^2) (a^2 + b^2)} \qquad (26),$$

$$\delta = \frac{(a + b)^2 (1 + b)^2 \{3a^2 + a^3 + (1 + 3a) b^2\}}{(1 - a)^3 b^2} \qquad (27),$$

$$C_0 = \frac{(a + b)^4 (1 + b)^2 \{3a^2 + a^3 + (1 + 3a) b^2\}}{(1 - a)^3 b^2 (a - b)^2} \qquad (28).$$

$$C_1^2 + 2C_0 = x_1^2 + x_2^2 = s_1 + s_2 - 2s_3$$

$$= m^2 \frac{1 - 4 (1 - 2m) \alpha - 8\alpha^2}{4 (\alpha - m)^2} \qquad (29),$$

$$C^2 = \frac{C_1^2}{C_0} = -2 - c^2 \frac{1 - 4 (1 - 2m) \alpha - 8\alpha^2}{4\alpha (\alpha - m)}$$

$$= -2 - \frac{1 - c^4}{c^2} \left\{\frac{1}{4\alpha} - 1 - 2 (\alpha - m)\right\}$$

$$= \left(\frac{1}{c} - c\right)^2 + \left(\frac{1}{c^2} - c^2\right) \frac{-1}{4\alpha}$$

$$= \frac{16a^2b^2}{(a^2 - b^2)^2} + \frac{8ab (a^2 + b^2) 2ab (1 - a) (a^2 + b^2)}{(a^2 - b^2)^3 \{3a^2 + a^3 + (1 + 3a) b^2\}}$$

$$= \frac{64a^3b^2 (a^3 - b^4)}{(a^2 - b^2)^3 \{3a^2 + a^3 + (1 + 3a) b^2\}} \qquad (30),$$

$$C_1^2 = \frac{16a^3 (a^3 - b^4) (1 + b)^2}{(1 - a)^3 (a - b)^6} \qquad (31),$$

$$\tfrac{1}{4} (\tfrac{1}{o} - o)^2 = \tfrac{1}{4} C^2 = \frac{16a^3b^2 (a^3 - b^4)}{3a^8 + a^9 + 6 (a^4 - a^5) b^4 - (1 + 3a) b^8 - 8a^3b^2 (a^3 - b^4)} \qquad (32),$$

$$\tfrac{1}{4} (\tfrac{1}{o} + o)^2 = \frac{3a^8 + a^9 + 6 (a^4 - a^5) b^4 - (1 + 3a) b^8 + 8a^3b^2 (a^3 - b^4)}{3a^8 + a^9 + 6 (a^4 - a^5) b^4 - (1 + 3a) b^8 - 8a^3b^2 (a^3 - b^4)} \qquad (33).$$

Thus in the quadric transformation

$$\lambda = \frac{2}{\frac{1}{o} + o}, \quad \lambda' = \frac{\frac{1}{o} - o}{\frac{1}{o} + o} \qquad (34),$$

$$\tfrac{1}{4}\left(\frac{1}{\lambda} - \lambda\right)^2 = \frac{64a^6b^4 (a^3 - b^4)^2}{D} \qquad (35),$$

$$\tfrac{1}{4}\left(\frac{1}{\lambda} + \lambda\right)^2 = \frac{\{3a^8 + a^9 + 6 (a^4 - a^5) b^4 - (1 + 3a) b^8\}^2}{D} \qquad (36),$$

$$D = (a^4 - b^4)^3 \{(3a^2 + a^3)^2 - (1 + 3a)^2 b^4\} \qquad (37),$$

involving powers of b^4 only; this will be useful in the sequel.

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To connect up with the order $\mu' = 4n + 2 = \frac{1}{2}\mu$, denoting $p(\mu')$ by p',

$$\frac{p'+1}{p'-1} = \sqrt{\frac{s(\omega_3) - s(6v)}{s(\omega_3) - s(3v)}} = \frac{(1 - 2m + 2m^2)\beta - (1 - 3m + 3m^2)}{m(1-m)(2\beta - 1)}$$

$$= -\frac{(p^2+1)\epsilon - 2p^2}{(p^2-1)\epsilon} = -\frac{1+b^2}{1-b^2} \cdot \frac{(a-b^2)^2}{2a^3 + a^4 - (1+2a)b^4} \quad (38),$$

$$p' = \frac{p^2 - \epsilon}{(\epsilon - 1)p^2} = -\frac{(1 + b^2 \,.\, a^2 - b^2)^2}{1 - b^4 \,.\, 2a^3 + a^4 - (1+2a)b^4} \quad (39).$$

We have found

$$\underset{o}{\mathrm{dn}} f\mathrm{K}' = c = \frac{a-b}{a+b} = \frac{\Theta(1-f)\mathrm{K}'}{\Theta(f\mathrm{K}')} \quad (40),$$

and proceeding with the series, writing f for $f\mathrm{K}'$,

$$\underset{o}{\mathrm{dn}} 3f = \frac{1-m}{m} c = \frac{1-b}{1+b} \quad (41),$$

$$\underset{o}{\mathrm{dn}} (2r+1) f = \frac{\mathrm{N}_{2r+1}}{\mathrm{D}_{2r+1}} c \quad (42).$$

Working these out with the assistance of the analysis given subsequently,

$$\underset{o}{\mathrm{dn}} 5f = \frac{a^3 - (1 + a - a^2)b^4 + ab(a - a^2 - a^3 + b^4)}{a^3 - (1 + a - a^2)b^4 - ab(a - a^2 - a^3 + b^4)} \quad (43),$$

and generally

$$\underset{o}{\mathrm{dn}} (2r+1) f = \frac{\mathrm{E}_{2r+1} + b\mathrm{F}_{2r+1}}{\mathrm{E}_{2r+1} - b\mathrm{F}_{2r+1}} \quad (44),$$

where F is a rational function of a and b^4, and E derivable from it by the substitution

$$\left(a, \frac{1}{a}\right)\left(b, \frac{1}{b}\right).$$

Thus for

$$\mu = 8n + 4, \quad \underset{o}{\mathrm{dn}} (2n+1) f = 1, \quad \mathrm{F}_{2n+1} = 0 \quad (45).$$

The results in the sequel give

$$\mathrm{F}_7 = a^4(1 + a - 2a^2 - a^3) - (a + a^2 - a^3 - 3a^4)b^4 - a^2 b^8 \quad (46),$$

and therefore

$$\mathrm{E}_7 = a^5 - (3a^3 + a^4 - a^5 - a^6)b^4 + (1 + 2a - a^2 - a^3)b^8 \quad (47),$$

$$\mathrm{F}_9 = a^6(3 + 0 - 3a^2 - a^3) - 3a^3(1 + 2a - 2a^2 - 2a^3)b^4$$
$$+ (1 + 3a + 0 - 6a^3 + 0 + 0 - a^6)b^8 + a^3 b^{12} \quad (48),$$

$$E_9 = a^6 + a^3(-1 + 0 + 0 - 6a^3 + 0 + 3a^5 + a^6)\, b^4$$
$$+ 3a^3(2 + 2a - 2a^2 - a^3)\, b^8 - (1 + 3a + 0 - 3a^3)\, b^{12} \qquad (49),$$

$$F_{11} = A_0 + A_1 b^4 + A_2 b^8 + A_3 b^{12} + A_4 b^{16} + A_5 b^{20} \qquad (50),$$

$$E_{11} = B_5 + B_4 b^4 + B_3 b^8 + B_2 b^{12} + B_1 b^{16} + B_0 b^{20} \qquad (51),$$

where the A's are found in § 45 (3), and the B's are derivable by the substitution $\left(a, \dfrac{1}{a}\right)$.

These expressions have been tested by putting

$$F_5 = 0, \quad b^4 = -a + a^2 + a^3 \qquad (52),$$

which makes

$$E_7 = a^2(1 - a^2)^3, \qquad F_7 = a(1 - a^2)^3 \qquad (53),$$

$$E_9 = a^3(1 + a)^5(1 - a)^4, \quad F_9 = a^2(1 + a)^5(1 - a)^7 \qquad (54),$$

$$E_{11} = a^5(1 + a)^8(1 - a)^7, \quad F_{11} = a^4(1 + a)^8(1 - a)^7 \qquad (55),$$

and thus verifies

$$\frac{\mathrm{dn}7f}{o} = \frac{o}{\mathrm{dn}3f} = \frac{1 + b}{1 - b}, \quad \frac{\mathrm{dn}9f}{o} = \frac{\mathrm{dn}11f}{o} = \frac{a + b}{a - b} \qquad (56).$$

42. $\mu = 20, \quad f = \frac{1, 3, 7, 9}{10}.$

$$b^4 = -a + a^2 + a^3, \quad a^3 - b^4 = a(1 - a) \qquad (1),$$

$$C^2 = \frac{+64a^2b^2}{-(1 + a^2)(1 + 2a - 6a^2 - 2a^3 + a^4) - 8a^2b^2} \qquad (2),$$

$$C_1^2 = \frac{16a^4(1 + b)^2}{(1 - a)^2(a - b)^6} \qquad (3),$$

$$\tfrac{1}{4}\left(\frac{1}{\lambda} - \lambda\right)^2 = \frac{64a^4(-a + a^2 + a^3)}{(1 - a^2)^5(1 + 4a - a^2)} \qquad (4),$$

$$\tfrac{1}{4}\left(\frac{1}{\lambda} + \lambda\right)^2 = \frac{(1 + a^2)^2(1 + 2a - 6a^2 - 2a^3 + a^4)^2}{(1 - a^2)^5(1 + 4a - a^2)} \qquad (5),$$

Now, as in the 'Archiv,' III., 1, p. 75, with the b there replaced by $-\dfrac{1 + a}{1 - a}$, and $\sqrt[4]{B}$ by $\dfrac{2b}{1 - a}$,

$$\mathrm{I}(v) = \int_x^0 \frac{-\dfrac{\mathrm{P}(v)}{\mathrm{M}}(\mathrm{D}-x^2) + \dfrac{\mathrm{Q}(v)}{\mathrm{M}^3}}{\mathrm{D}-x^2} \frac{dx}{\sqrt{\mathrm{X}_1\mathrm{X}_2}},$$

$$= \tfrac{1}{5}\cos^{-1}\frac{\mathrm{R}_0 + \mathrm{R}_1x + \mathrm{R}_2x^2 + \mathrm{R}_3x^3 + x^4}{(\mathrm{D}-x^2)^{\frac{5}{2}}}\sqrt{\tfrac{1}{2}\mathrm{X}_1},$$

$$= \tfrac{1}{5}\sin^{-1}\frac{\mathrm{R}_0 - \mathrm{R}_1x + \mathrm{R}_2x^2 - \mathrm{R}_3x^3 + x^4}{(\mathrm{D}-x^2)^{\frac{5}{2}}}\sqrt{\tfrac{1}{2}\mathrm{X}_2} \tag{6}$$

$$\mathrm{X}_1,\quad \mathrm{X}_2 = 1 \pm \mathrm{C}x - x^2 \tag{7}$$

$$\mathrm{D} = c^2 = \left(\frac{a-b}{a+b}\right)^2 = \frac{\mathrm{dn}^2 f\mathrm{K}'}{o^2},\quad \mathrm{R}_0 = c^5 \tag{8}$$

$$\frac{\mathrm{zn}\,\tfrac{1}{2}\mathrm{K}'}{o} = \frac{\mathrm{P}(5v)}{\sqrt{\mathrm{C}_0}} = \tfrac{1}{2}\left(\tfrac{1}{o} - o\right) \tag{9}$$

$$\mathrm{P}(5v) = \tfrac{1}{2}\mathrm{C}_1 = \frac{2a^2(1+b)}{(1-a)(a-b)^3} \tag{10}$$

$$\mathrm{C}_0 = \frac{a(1+3a-a^2-a^3-2ab^2)(a^2-b^2)^2}{4(1-a)^2b^2} \tag{11}$$

$$\frac{\mathrm{Q}(v)}{\mathrm{M}^3} = \frac{c^3}{\sqrt{\alpha(\alpha-m)}} \tag{12}$$

$$\mathrm{R}_3\sqrt{\mathrm{C}_0} = -5\mathrm{P}(v) + \mathrm{P}(5v) = -1 - \frac{y^2}{y-x} = -\frac{2a}{a-b} - \frac{1+b}{1-a} \tag{13}$$

while R_1 and R_2 can be calculated and the whole result verified by differentiating (6) and equating coefficients of powers of x.

A change of b into $-b$ will change v into $9v$.

The c employed for $\mu = 10$ is $\dfrac{1}{p'}$, so that, putting

$$\frac{1-c}{1+c} = \frac{p'-1}{p'+1} = a' \tag{14}$$

we find

$$a' = -\frac{(a-b^2)^2}{a(1+a)^2} = \frac{1-2a-a^2+2b^2}{(1+a)^2} \tag{15}$$

so that, with

$$u = \int_a^\infty \frac{da}{\sqrt{\mathrm{A}}},\quad \mathrm{A} = 4b^4 \tag{16}$$

$$a = a(u),\quad a' = a(u - \tfrac{1}{3}\omega) \tag{17}$$

(Kiepert, 'Math. Ann.,' 32, p. 107), and then, with $a = a_{20}$,

$$\eta_1 = a + 1 - \frac{1}{a},$$

$$\eta_2 = \frac{a^2 - 1}{a^2 - 4a - 1},$$

$$\eta_3 = -\frac{a^2 - 1}{4a}, \text{ \&c.} \qquad (18).$$

Then, in § 28,

$$\left(\frac{\Theta \frac{1}{10} K'}{\Theta 0}\right)^{20} = c_1^{20} = \frac{m\alpha^7 (\alpha - m)^6 \gamma_9}{x^{\frac{1}{3}} \gamma_{11}}. \text{ \&c.} \qquad (19).$$

43. $\mu = 28,\ v = K + \frac{rK'i}{14},\ r = 1, 3, 5, 9, 11, 13.$

$$D_7 - cN_7 = 0 \qquad (1).$$

$$\frac{a}{b} = \frac{N_7 + D_7}{N_7 - D_7} \qquad (2).$$

A change of b into $-b$ changes v into $13v$.

Now, with $1 - 2m = p,\ p\alpha = m(1 - m)\beta,\ 8p\alpha = (1 - p^2)(\epsilon + 1)$,

$$D_7 = m^2 (1 - m)^2 \left\{ p \left(\frac{\epsilon + 1}{2}\right)^2 - \frac{3p - 1}{2} \frac{\epsilon + 1}{2} - \left(\frac{p - 1}{2}\right)^2 \right\}$$

$$= \tfrac{1}{4} m^2 (1 - m)^2 \{- p^2 + \epsilon(\epsilon - 1) p + \epsilon\} \qquad (3);$$

and changing p into $-p$,

$$N_7 = \tfrac{1}{4} m^2 (1 - m)^2 \{- p^2 - \epsilon(\epsilon - 1) p + \epsilon\} \qquad (4).$$

$$\frac{N_7 + D_7}{N_7 - D_7} = \frac{\epsilon - p^2}{-\epsilon(\epsilon - 1) p} = \frac{a}{b} \qquad (5).$$

$$ap\epsilon(\epsilon - 1) + b(\epsilon - p^2) = 0 \qquad (6).$$

Substituting for p and ϵ in terms of a and b, and cancelling a factor $(a^2 - b^2)^2$, there results

$$ab^8 + (1 + a - a^2 - 3a^3) b^4 - a^3 (1 + a - 2a^2 - a^3) = 0 \qquad (7),$$

a quadratic for b^4.

To obtain a rational expression for C, we put

$$a^3 - b^4 = a^3 (1 - a) c^2 \qquad (8),$$

and now the relation (7) becomes

$$(a^2c^2 + a)^2 - (1 + a)^2 c^2 = 0 \qquad (9),$$

$$a^2c^2 + ac + a + c = 0 \tag{10}$$

$$c = \frac{-1 - a + \sqrt{A}}{2a^2} \tag{11}$$

$$A = (1 + a)^2 - 4a^3 = (1 - a)(1 + 3a + 4a^2) \tag{12}$$

so that, with

$$a = a(u), \quad u = \int_a^\infty \frac{da}{\sqrt{A}} \tag{13}$$

then

$$c = a(u - \tfrac{1}{3}\omega) \tag{14}$$

(Kiepert, 'Math. Ann.,' 32, p. 108).

Introducing these values into (35), § 41,

$$\tfrac{1}{4}\left(\frac{1}{\lambda} - \lambda\right)^2 = \frac{-128a^7(1 + a - 2a^2 - a^3)}{(1 + a)^7(1 - a)^3[7a(1 - a^2) + (2 - a + a^2)\sqrt{A}]} \tag{15}$$

to be compared with the expression for ξ_2 in $\mu = 14$, L.M.S., 27, p. 440, putting

$$c = \frac{1 - a}{2a}, \quad 1 + 2c = \frac{1}{a} \tag{16}$$

Writing q or p' for p_{14},

$$q = p' = \frac{p^2 - \epsilon}{(\epsilon + 1)p^2}$$

$$= -\frac{-a^3 + (-a^2 + a^3 + a^4)b^2 + (1 + a - a^2)b^4 - ab^6}{(1 + a)(a - b^2)(a^2 - b^4)} \tag{17}$$

and after reduction

$$(1 + a)^2q^2 - a(1 + 3a - \sqrt{A})q$$
$$+ a^2(1 - a) - a\sqrt{A} = 0 \tag{18}$$

or

$$q^2 - a(t + 1)q + at = 0, \quad t = a(u - \tfrac{2}{3}\omega) \tag{19}$$

to be interpreted as an elliptic function relation.

Rationalising again

$$(1 + a)^2q^4 - 2a(1 + 3a)q^3 + 2a^2(1 + a)q^2$$
$$+ 2a^2(1 - a)q - a^2(1 - a^2) = 0 \tag{20}$$

or arranged in powers of a,

$$a^4 + 2q(q - 1)a^3 + (q^4 \pm 6q^3 + 2q^2 \mp 2q - 1)a^2$$
$$+ 2q^3(q - 1)a + q^4 = 0 \tag{21}$$

$$a^2 + q(q - 1)a + q^2 = a\sqrt{Q} \tag{22}$$

$$Q = 4q^3 + (q - 1)^2 \tag{23}$$

44. $$\mu = 36, \quad v = \mathrm{K} + \frac{2r+1}{18}\mathrm{K}'i.$$

The relation

$$\{s(\omega) - s(9v)\}^2 = (s_1 - s_3)(s_2 - s_3) \qquad (1)$$

leads to

$$\frac{\mathrm{N}_9}{\mathrm{D}_9} = \frac{1+b}{1-b} \qquad (2),$$

$$b = \frac{\mathrm{N}_9 + \mathrm{D}_9}{\mathrm{N}_9 - \mathrm{D}_9} = \frac{\epsilon^2[(\epsilon - 2)p^2 + \epsilon]}{p[p^2 - \epsilon(\epsilon^2 - \epsilon + 1)]} \qquad (3),$$

and putting

$$p^2 - 1 = q, \quad \frac{\epsilon - 1}{q} = r, \quad 2r - 1 = s, \quad \frac{\epsilon - 1}{s} = t, \quad t + 1 = u \qquad (4),$$

$$\frac{2\epsilon^2}{bp} + \epsilon + 1 - \epsilon\frac{\epsilon - 1}{u} = 0 \qquad (5),$$

and thereby the superfluous factor $a^2 - b^2$ is cancelled, and finally

$$a^3b^{12} + (1 + 3a + 0 - 6a^3 + 0 + 0 - a^6)\,b^8$$
$$- 3(1 + 2a - 2a^2 - 2a^3)\,a^3b^4 + (3 + 0 - 3a^2 - a^3)\,a^6 = 0 \qquad (6),$$

a cubic equation for b^4.

Putting as before

$$a^3 - b^4 = \frac{a^3(1-a)}{(c+1)^2} \qquad (7),$$

this relation (63) becomes

$$[c^3 + 2(1 - a^2)c]^2 - (1-a)^2[(2+a)c^2 + (1+a)^2]^2 = 0 \qquad (8),$$

$$c^3 + (1-a)(2+a)c^2 + 2(1-a^2)c + (1-a)(1+a)^2 = 0 \qquad (9),$$

and putting $c = (1 + a)q$,

$$(1+a)q^3 - (1-a)(2+a)q^2 + 2(1-a)q + 1 - a = 0 \qquad (10),$$

and arranged as a quadratic in a,

$$q^2a^2 - (q^3 - q^2 - 2q - 1)a - (q+1)(q^2 + q + 1) = 0 \qquad (11),$$

$$a = \frac{q^3 - q^2 - 2q - 1 + \sqrt{\mathrm{Q}}}{2q^2} \qquad (12),$$

$$\mathrm{Q} = q^6 + 2q^5 + 5q^4 + 10q^3 + 10q^2 + 4q + 1 \qquad (13),$$

as in $\mu = 18$, § 32; and

$$c = \frac{q^3 + q^2 - 2q - 1 + \sqrt{\mathrm{Q}}}{2q} \qquad (14).$$

45. $\mu = 44, \quad v = \mathrm{K} + f\mathrm{K}'i, \quad f = \frac{2r+1}{22}.$

Here the relation to be reduced becomes

$$\frac{b}{a} = \frac{\mathrm{N}_{11} + \mathrm{D}_{11}}{\mathrm{N}_{11} - \mathrm{D}_{11}} = \frac{\epsilon[(\epsilon - 2)p^2 + \epsilon][(\epsilon^2 - \epsilon - 1)p^2 + \epsilon]}{p[p^2 - \epsilon^2][p^2 - \epsilon^3 + \epsilon^2 - \epsilon]} \qquad (1).$$

The algebraical work has been carried out by Mr. R. H. MACMAHON, and he reduces the relation to a quintic in b^4,

$$\mathrm{A}_0 + \mathrm{A}_1 b^4 + \mathrm{A}_2 b^8 + \mathrm{A}_3 b^{12} + \mathrm{A}_4 b^{16} + \mathrm{A}_5 b^{20} = 0 \qquad (2),$$

$$\begin{aligned}
\mathrm{A}_0 &= a^{10} \quad (1 - 2a - 5a^2 + 2a^3 + 4a^4 + a^5),\\
\mathrm{A}_1 &= a^7(-1 + 2 + 15 + 8 - 19 - 10a^5),\\
\mathrm{A}_2 &= a^6(-15 - 27 + 14 + 45 + 0 - 11 + 0 + 3 + a^8),\\
\mathrm{A}_3 &= a^3 \quad (6 + 22 + 5 - 40 - 23 + 26 + 5 - 8 - 3a^8),\\
\mathrm{A}_4 &= -1 - 5a - 5a^2 + 9 + 13 - 5 - 10 + 6 + 3a^8,\\
\mathrm{A}_5 &= -a^5
\end{aligned} \qquad (3).$$

As before, putting

$$a^3 - b^4 = \frac{a^3(1-a)}{(v+1)^2}, \text{ and } v = (1+a)c \qquad (4),$$

the relation can be halved in degree, and becomes

$$\begin{aligned} c^5a^4 + c^2(c+1)(2c^2+2c+1)a^3 + (c^5 + 3c^4 + 6c^3 + 8c^2 + 4c + 1)a^2 \\ - c^2(2c^2 + 5c + 3)a - (c+1)^4 = 0 \end{aligned} \qquad (5),$$

a quartic in a of similar structure to the one in $\mu = 22$, § 33 (3), and capable of resolution in a similar manner into

$$[2c^3a^2 + (c+1)(2c^2+2c+1)a - (c+1)]^2 = \mathrm{C}[(c-1)a + (c+1)]^2 \qquad (6),$$

$$\mathrm{C} = 4c(c+1)^2 + 1 \qquad (7).$$

Put

$$\frac{2c^3a^2 + (c+1)(2c^2+2c+1)a - (c+1)}{(c-1)a + (c+1)} = 2c^2x - 2c - 1 \qquad (8),$$

and then

$$x = \frac{2c + 1 + \sqrt{\mathrm{C}}}{2c^2} = c(u - \tfrac{4}{5}\omega) \qquad (9),$$

if

$$c = c(u), \quad u = \int_c^\infty \frac{dc}{\sqrt{\mathrm{C}}} \qquad (10);$$

and

$$cx = \frac{c^2a^2 + (c^2 + 3c + 1)a + c + 1}{(c-1)a + (c+1)} \qquad (11),$$

a relation which requires interpretation, connecting c, x, and a, elliptic functions of u.

It is now possible theoretically to determine the coefficients in

$$\mathrm{I}(v) = \int_x^0 \frac{-\frac{\mathrm{P}(v)}{\mathrm{M}}(\delta - x^2) + \frac{\mathrm{Q}(v)}{\mathrm{M}^3}}{\delta - x^2} \frac{dx}{\sqrt{\mathrm{X}_1 \mathrm{X}_2}}$$

$$= \frac{1}{11} \cos^{-1} \frac{\mathrm{R}_0 + \mathrm{R}_1 x + \ldots + \mathrm{R}_9 x^9 - x^{10}}{(\delta - x^2)^{\frac{11}{2}}} \sqrt{\tfrac{1}{2}\mathrm{X}_1}$$

$$= \frac{1}{11} \sin^{-1} \frac{\mathrm{R}_0 - \mathrm{R}_1 x + \ldots - \mathrm{R}_9 x^9 - x^{10}}{(\delta - x^2)^{\frac{11}{2}}} \sqrt{\tfrac{1}{2}\mathrm{X}_2} \qquad (12),$$

and to construct an algebraical herpolhode and associated top motion complete in 44 branches.

This is as far as we can go at present, as the next cases of $\mu = 52$ and 60 must await the solution of $\mu = 26$ and 30, not yet accomplished.

46. II. $\quad \frac{1}{4}\mu = 2n, \quad \mu = 8n, \quad v = \mathrm{K} + f\mathrm{K}'i, \quad f = \frac{2r+1}{4n}$.

$$\mathrm{X}_1 = \mathrm{C}_0 + \mathrm{C}_1 x + x^2 = (x_1 + x)(x_2 + x) \qquad (1),$$

$$\mathrm{X}_2 = \mathrm{C}_0 - \mathrm{C}_1 x + x^2 = (x_1 - x)(x_2 - x) \qquad (2),$$

$$\mathrm{C}_0 = s(\omega_3) - s(2nv) = \sqrt{(s_1 - s_3 \,.\, s_2 - s_3)} \qquad (3),$$

$$\mathrm{C}_1 = \sqrt{(s_1 - s_3)} + \sqrt{(s_2 - s_3)} = 2\mathrm{P}(2nv) \qquad (4),$$

$$\mathrm{C}_0\mathrm{C}_1 = \tfrac{1}{2}i\wp'(2nv) = \mathrm{Q}(2nv) \qquad (5),$$

$$\left(\tfrac{1}{o} + o\right)^2 = \frac{\mathrm{C}_1{}^2}{\mathrm{C}_0} = \mathrm{C}^2 \qquad (6),$$

and normalised by $\mathrm{M} = \sqrt{\mathrm{C}_0}$,

$$\mathrm{X}_1, \mathrm{X}_2 = 1 \pm \mathrm{C}x + x^2, \quad x_1 = \tfrac{1}{o}, \quad x_2 = o \qquad (7),$$

$$x = o\,\mathrm{sn}(\mathrm{K} - mt), \quad \frac{\mathrm{dn}^2 f\mathrm{K}'}{o^2} = \mathrm{D} \qquad (8),$$

$$\frac{\mathrm{zn} f\mathrm{K}'}{o} = \frac{\mathrm{P}(v)}{\mathrm{M}} \qquad (9),$$

and the associated integral

$$\mathrm{I}(v) = \int_x^0 \frac{-\frac{\mathrm{P}(v)}{\mathrm{M}}(\mathrm{D} - x^2) + \frac{\mathrm{Q}(v)}{\mathrm{M}^3}}{\mathrm{D} - x^2} \frac{dx}{\sqrt{(\mathrm{X}_1 \mathrm{X}_2)}}$$

$$= \frac{1}{2n} \cos^{-1} \frac{\mathrm{R}_0 + \mathrm{R}_1 x + \ldots + \mathrm{R}_{2n-2} x^{2n-2} + \mathrm{R}_{2n-1} x^{2n-1}}{(\mathrm{D} - x^2)^n} \sqrt{\tfrac{1}{2}\mathrm{X}_1}$$

$$= \frac{1}{2n} \sin^{-1} \frac{\mathrm{R}_0 - \mathrm{R}_1 x + \ldots + \mathrm{R}_{2n-2} x^{2n-2} - \mathrm{R}_{2n-1} x^{2n-1}}{(\mathrm{D} - x^2)^n} \sqrt{\tfrac{1}{2}\mathrm{X}_2} \qquad (10)$$

$$R_0{}^2 = D^{2n}, \quad R_{2n-2} = -[2nP(v) + P(2nv)]\,R_{2n-1}, \quad R_{2n-1} = (-1)^n \tag{11}$$

$$\frac{Q(v)}{M^3} = \left(\frac{1}{o^2} - o^2\right) \operatorname{sn} f K' \operatorname{cn} f K' \operatorname{dn} f K' \tag{12}$$

In this case II. put

$$(\alpha - m)(\alpha - m + 1) = \alpha^2 a^4 \tag{13}$$

then

$$C_0{}^2 = \frac{m^4 \alpha^4 a^4}{(\alpha - m)^4}, \quad C_0 = a^2 [s(\omega) - s(2v)] \tag{14}$$

$$\delta = s(v) - s(\omega_3) = -\frac{m^2 \alpha}{\alpha - m} \text{ (positive)} \tag{15}$$

Now, with

$$1 - 2m = p, \quad (1 - 2m)\alpha = m(1 - m)\beta = m(1 - m)\frac{\epsilon + 1}{2} \tag{16}$$

$$\frac{s(\omega) - s(4v)}{s(\omega) - s(2v)} = \left(\frac{\beta - 1}{\beta}\right)^2 = \left(\frac{\epsilon - 1}{\epsilon + 1}\right)^2 = b^2, \text{ suppose} \tag{17}$$

$$\epsilon = \frac{1 - b}{1 + b}, \quad \beta = \frac{1}{1 + b} \tag{18}$$

and the relation (13) becomes

$$1 - a^4 + \frac{p}{\alpha} - \frac{1 - p^2}{4\alpha^2} = 0 \tag{19}$$

$$1 - a^4 + \frac{4p^2(1 + b)}{1 - p^2} - \frac{4p^2(1 + b)^2}{1 - p^2} = 0 \tag{20}$$

$$p^2 = \frac{1 - a^4}{(1 + 2b)^2 - a^4}, \quad m - m^2 = \frac{1 - p^2}{4} = \frac{b(1 + b)}{(1 + 2b)^2 - a^4} \tag{21}$$

$$p\alpha = \tfrac{1}{4}(1 - p^2)\beta = \frac{b}{(1 + 2b)^2 - a^4} \tag{22}$$

$$\frac{p}{\alpha} = \frac{p^2}{p\alpha} = \frac{1 - a^4}{b} \tag{23}$$

$$\frac{1}{\alpha^2} = \frac{p^2}{p^2\alpha^2} = \frac{(1 - a^4)[(1 + 2b)^2 - a^4]}{b^2} \tag{24}$$

$$\begin{aligned}
(\tfrac{1}{o} + o)^2 = C^2 &= \frac{C_1{}^2}{C_0} \\
&= 2 + \frac{1 - 4(1 - 2m)\alpha - 8\alpha^2}{4\alpha^2 a^2} \\
&= 2 + \frac{1}{4\alpha^2 a^2} - \frac{p}{\alpha a^2} - \frac{2}{a^2} \\
&= 2 + \frac{(1 - a^4)[(1 + 2b)^2 - a^4]}{4a^2 b^2} - \frac{1 - a^4}{a^2 b} - \frac{2}{a^2} \\
&= \frac{(1 - a^2)^2[(1 + a^2)^2 - 4b^2]}{4a^2 b^2}
\end{aligned} \tag{25}$$

$$(\tfrac{1}{o} - o)^2 = \frac{(1+a^2)^2[(1-a^2)^2 - 4b^2]}{4a^2b^2} \qquad (26),$$

$$1 - 8(1-2m)\alpha = \frac{(1-2b)^2 - a^4}{(1+2b)^2 - a^4} \qquad (27).$$

We now find, as in § 41, writing f for $f\mathrm{K}'$,

$$\frac{\mathrm{dn}^2 f}{o^2} = \frac{\sigma - s_3}{\mathrm{C}_0} = \frac{1}{a^2}\left(\frac{m}{\alpha} - 1\right)$$

$$= \frac{\sqrt{1+a^2}\sqrt{2b+1-a^2} - \sqrt{1-a^2}\sqrt{2b+1+a^2}}{\sqrt{1+a^2}\sqrt{2b+1-a^2} + \sqrt{1-a^2}\sqrt{2b+1+a^2}} \qquad (28),$$

$$\frac{\mathrm{dn} f}{o} = \frac{\sqrt{1+a^2}\sqrt{2b+1-a^2} - \sqrt{1-a^2}\sqrt{2b+1+a^2}}{2a\sqrt{b}} \qquad (29),$$

and then, from the relation

$$\frac{\mathrm{dn}\,(2n+1)f}{\mathrm{dn} f} = \frac{\mathrm{N}_{2n+1}}{\mathrm{D}_{2n+1}} \qquad (30),$$

writing c for a^2, and putting

$$(1+c)(2b+1-c) = \mathrm{B}_1, \quad (1-c)(2b+1+c) = \mathrm{B}_2 \qquad (31),$$

$$\frac{\mathrm{dn} 3f}{o} = \frac{(b+c)\sqrt{\mathrm{B}_1} - (b-c)\sqrt{\mathrm{B}_2}}{2(b+1)\sqrt{bc}} \qquad (32),$$

$$\frac{\mathrm{dn} 5f}{o} = \frac{(b^3 - b^2c - bc^2 + c)\sqrt{\mathrm{B}_1} - (b^3 + b^2c - bc^2 - c)\sqrt{\mathrm{B}_2}}{2(b^3 - b^2 - b + c^2)\sqrt{bc}} \qquad (33),$$

$$\frac{\mathrm{dn} 7f}{o} = \frac{\mathrm{E}_7\sqrt{\mathrm{B}_1} - \mathrm{F}_7\sqrt{\mathrm{B}_2}}{2\{(b^2-1)(b^4-c^2) + b(b^2-c^2)(2b^2-1-c^2)\}\sqrt{bc}} \qquad (34),$$

$$\mathrm{E}_7 = b^6 + 2b^5c - b^4c^2 - c(3+c^2)b^3 - b^2c^2 + c(1+c^2)b + c^4$$
$$= (b^2-c^2)^2(b^2+c^2) + bc(b^2-1)(2b^2-1-c^2),$$

and F_7 is obtained from E_7 by changing c into $-c$.

47. $\mu = 16, \quad v = \mathrm{K} + f\mathrm{K}'i, \quad f = \frac{1,\,3,\,5,\,7}{8},$

and now the condition

$$s(\omega) - s(4v) = \sqrt{(s_1 - s_3,\, s_2 - s_3)} = \mathrm{C}_0 \qquad (1)$$

reduces to

$$a = b \qquad (2),$$

$$\tfrac{1}{o} + o = \mathrm{C} = \tfrac{1}{2}\left(b - \frac{1}{b}\right)^2 \tag{3}$$

$$\tfrac{1}{o} - o = \frac{(b^2+1)\sqrt{(b^4 - 6b^2 + 1)}}{2b^2} \tag{4}$$

$$\mathrm{D} = \frac{\mathrm{dn}^2 f}{o^2} = \frac{[\sqrt{(b^2+1)}\sqrt{(b^2-2b-1)} - (b+1)\sqrt{(b^2-1)}]^2}{4b^3} \tag{5}$$

$$\begin{aligned} \mathrm{I}(v) &= \tfrac{1}{4}\cos^{-1}\frac{\mathrm{R}_0 + \mathrm{R}_1 x + \mathrm{R}_2 x^2 + x^3}{(\mathrm{D} - x^2)^2}\sqrt{\tfrac{1}{2}\mathrm{X}_1}, \\ &= \tfrac{1}{4}\sin^{-1}\frac{\mathrm{R}_0 - \mathrm{R}_1 x + \mathrm{R}_2 x^2 - x^3}{(\mathrm{D} - x^2)^2}\sqrt{\tfrac{1}{2}\mathrm{X}_2} \end{aligned} \tag{6}$$

$$\mathrm{X}_1, \mathrm{X}_2 = 1 \pm \tfrac{1}{2}\left(b - \frac{1}{b}\right)^2 x + x^2 \tag{7}$$

$$\mathrm{R}_0 = \mathrm{D}^2 \tag{8}$$

$$\mathrm{R}_1 = \frac{\begin{Bmatrix} -b^8 + 0 - 4 - 4 - 2 + 4 + 6 + 4 + 1 - (b+1)^2 \\ (b^3 - b^2 - b - 1)\sqrt{(b^4-1)(b^2-2b-1)} \end{Bmatrix}}{2b^4} \tag{9}$$

$$\mathrm{R}_2 = -\frac{2b + 1 + (b+1)\sqrt{(b^4-1)(b^2-2b-1)}}{b^2} \tag{10}$$

$$\frac{\mathrm{P}(v)}{\mathrm{M}} = \frac{\mathrm{zn} f \mathrm{K}'}{o} = \frac{4(b+1)\sqrt{(b^4-1)(b^2-2b-1)} - (b^2+2b+3)(b^2-2b-1)}{16b^2} \tag{11}$$

$$\frac{\mathrm{Q}(v)}{\mathrm{M}^3} = \frac{(b+1)\sqrt{(b^4-1)(b^2-2b-1)}}{2b^2}\mathrm{D} \tag{12}$$

Bisection formulas for $\mu = 8$ in the region $v = \frac{1}{4}\mathrm{K}'i$ will lead to the same results, with $a_8 = a$,

$$\sqrt{a} = \frac{\sqrt{b^2 + 2b - 1} + \sqrt{b^2 - 2b - 1}}{2b} \tag{13}$$

$$\frac{1}{\sqrt{a}} = \frac{\sqrt{b^2 + 2b - 1} - \sqrt{b^2 - 2b - 1}}{2} \tag{14}$$

$$b^2 = \frac{1 + a}{a - a^2} \tag{15}$$

having the substitution

$$\left(a, \ -\frac{1}{a}\right), \quad \left(b, \ -\frac{1}{b}\right) \tag{16}$$

Hence the following table of section-values for $\mu = 16$, holding in the region $\infty > b > \sqrt{2} + 1$; tested numerically by $b = \left(\frac{\sqrt{5}+1}{2}\right)^2$, $o = \frac{1}{2}$, $\kappa = \frac{1}{4}$, as on p. 278, region C :—

$\frac{1}{\sqrt{2}}\left(b+\frac{1}{b}\right)$	$\frac{1}{\sqrt{o}}+\sqrt{o}$
$\frac{1}{\sqrt{2}}\sqrt{\left(b^2-6+\frac{1}{b^2}\right)}$	$\frac{1}{\sqrt{o}}-\sqrt{o}$
$\frac{1}{2}\left(b-\frac{1}{b}\right)^2$	$\frac{1}{o}+o$
$\frac{b^2+1\pm\sqrt{(b^4-6b^2+1)}}{2\sqrt{2b}}$	$\frac{1}{\sqrt{o}},\ \sqrt{o}$
$\frac{\sqrt{(b^2+2\sqrt{2b}+1)}\pm\sqrt{(b^2-2\sqrt{2b}+1)}}{2\sqrt[4]{2}\sqrt{b}}$	$\frac{1}{\sqrt[4]{o}},\ \sqrt[4]{o}$
$\frac{(b+1)\sqrt{b^2-1}+\sqrt{(b^2+1)(b^2-2b-1)}}{2b^{\frac{3}{2}}}$	$\frac{\operatorname{dn}\frac{3}{8}K'}{o}$
$\frac{(b-1)\sqrt{b^2-1}+\sqrt{(b^2+1)(b^2-2b-1)}}{2b^{\frac{1}{2}}}$	$\frac{\operatorname{dn}\frac{1}{8}K'}{o}$
$\frac{(b+1)\sqrt{b^2+2b-1}-(b-1)\sqrt{b^2-2b-1}}{(b+1)\sqrt{b^2+2b-1}+(b-1)\sqrt{b^2-2b-1}}$	$\operatorname{dn}\frac{1}{4}K'$
$\frac{(b-1)\sqrt{b^2+2b-1}+(b+1)\sqrt{b^2-2b-1}}{2\sqrt{b^4-1}}$	$\operatorname{sn}\frac{1}{4}K'$
$\frac{(b+1)\sqrt{b^2+2b-1}-(b-1)\sqrt{b^2-2b-1}}{2\sqrt{b^4-1}}$	$\operatorname{cn}\frac{1}{4}K'$
$\frac{(b-1)\sqrt{b^2+2b-1}-(b+1)\sqrt{b^2-2b-1}}{(b-1)\sqrt{b^2+2b-1}+(b+1)\sqrt{b^2-2b-1}}$	$\operatorname{dn}\frac{3}{4}K'$
$\frac{(b-1)\sqrt{b^2+2b-1}-(b+1)\sqrt{b^2-2b-1}}{2\sqrt{b^4-1}}$	$\operatorname{cn}\frac{3}{4}K'$
$\frac{(b+1)\sqrt{b^2+2b-1}+(b-1)\sqrt{b^2-2b-1}}{2\sqrt{b^4-1}}$	$\operatorname{sn}\frac{3}{4}K'$
$\frac{b^2+1-\sqrt{b^4-6b^2+1}}{b^2+1+\sqrt{b^4-6b^2+1}}$	$\operatorname{dn}\frac{1}{2}K'$
$\frac{b^2+1+\sqrt{b^4-6b^2+1}}{2(b^2-1)}$	$\operatorname{sn}\frac{1}{2}K'$
$\frac{b^2+1-\sqrt{b^4-6b^2+1}}{2(b^2-1)}$	$\operatorname{cn}\frac{1}{2}K'$

$\dfrac{\sqrt{b^2-2b-1}}{16b^2}[4(b-1)\sqrt{b^4-1}+(3b^2-2b+1)\sqrt{b^2-2b-1}]$	$\dfrac{\mathrm{zn}\,\frac{1}{8}\mathrm{K}'}{o}$
$\dfrac{\sqrt{b^2-2b-1}}{16b^2}[4(b-1)\sqrt{b^4-1}-(3b^2-2b+1)\sqrt{b^2-2b-1}]$	$\dfrac{\mathrm{zn}\,\frac{7}{8}\mathrm{K}'}{o}$
$\dfrac{\sqrt{b^2-2b-1}}{16b^2}[4(b+1)\sqrt{b^4-1}+(b^2+2b+3)\sqrt{b^2-2b-1}]$	$\dfrac{\mathrm{zn}\,\frac{3}{8}\mathrm{K}'}{o}$
$\dfrac{\sqrt{b^2-2b-1}}{16b^2}[4(b+1)\sqrt{b^4-1}-(b^2+2b+3)\sqrt{b^2-2b-1}]$	$\dfrac{\mathrm{zn}\,\frac{5}{8}\mathrm{K}'}{o}$
$c_1^{32}=\dfrac{b^{25}\left[\dfrac{(b+1)\sqrt{b^2-1}+\sqrt{(b^2+1)(b^2-2b-1)}}{2b^{\frac{3}{2}}}\right]^{16}}{(b-1)^7(b+1)^{15}(b^2-2b-1)^4(b^2+1)^6}$	$\left(\dfrac{\Theta\,\frac{1}{8}\mathrm{K}'}{\Theta 0}\right)^{32}$
$c_2^{8}=\dfrac{b^9}{(b^2-1)^3(b^2+1)^2}$	$\left(\dfrac{\mathrm{H}\,\frac{1}{4}\mathrm{K}'}{\mathrm{HK}'}\right)^{8}$

48. $\mu=24,\quad v=\mathrm{K}+f\mathrm{K}'i,\quad f=\frac{1,\,5,\,7,\,11}{12}.$

$$s(\omega)-s(6v)=\mathrm{C}_0=\frac{m^2\alpha^2a^2}{(\alpha-m)^2} \qquad (1),$$

and this leads to

$$(1-2m+2m^2)\,\beta-(1-3m+3m^2)$$
$$+(2\beta-1)(m-m^2)\,a=0 \qquad (2),$$

$$m(1-m)[3-a-2(1-a)\beta]+\beta-1=0 \qquad (3),$$

$$(1+b)[a(1-b)+1+3b]-(1+2b)^2+a^4=0 \qquad (4),$$

$$a(1-b^2)-b^2+a^4=0 \qquad (5),$$

$$b^2=\frac{a+a^4}{1+a}=a-a^2+a^3 \qquad (6),$$

$$\left(\tfrac{1}{o}+o\right)^2=\frac{(1-a)^6(1+a)^2}{4a^2(a-a^2+a^3)} \qquad (7),$$

$$\left(\tfrac{1}{o}-o\right)^2=\frac{(1+a^2)^3(1-4a+a^2)}{4a^2(a-a^2+a^3)} \qquad (8),$$

derivable from the results for $\mu=12$ by putting $a_{12}=c$,

$$c+\frac{1}{c}=\frac{(a-1)^2}{a} \qquad (9),$$

and now

$$\operatorname{dn}\tfrac{1}{12}\mathrm{K}' = \frac{\sqrt{1+a^2}\sqrt{(1-a^2+2\sqrt{a-a^2+a^3})}-\sqrt{1-a^2}\sqrt{(1+a^2+2\sqrt{a-a^2+a^3})}}{2a\sqrt[4]{a-a^2+a^3}}, \text{ \&c.} \tag{10}$$

In KIEPERT's notation ('Math. Ann.,' 32, p. 116),

$$\sqrt{\xi} = \frac{1-a^2+\sqrt{a-a^2+a^3}}{1+a^2}, \quad \xi_1 = \frac{1-4a+a^2}{1+a^2}, \quad \xi_2 = \frac{a}{1-a+a^2}, \text{ \&c.} \tag{11}$$

Treating $\mu = 24$ by the trisection method of § 14 applied to $\mu = 8$, we put

$$x = z(1-2z), \quad y = \frac{z(1-2z)}{1-z}, \quad y+1 = \frac{1-2z^2}{1-z} \tag{12}$$

so that, putting $t = z^2p$ in the equation (8), § 14, it may be written

$$(\tfrac{1}{o}+o)^2 = \frac{(1-2z)^4}{4z^2(1-z)^2} = -\frac{p^3(3p+4)}{4(p+1)^3} \tag{13}$$

To agree with the notation in KLEIN-FRICKE, 'Modul-Functionen,' 1, p. 688, put

$$p+1 = \tfrac{1}{3}y \tag{14}$$

$$\lambda = \tfrac{1}{4}(\tfrac{1}{o}+o)^2 = \frac{(3-y)^3(1+y)}{16y^3}, \quad \lambda - 1 = \frac{(3+y)^3(1-y)}{16y^3} \tag{15}$$

$$\lambda(\lambda-1) = \frac{(9-y^2)^3(1-y^2)}{256y^6} = \frac{(8+x^3)^3x^3}{256(1-x^3)^3} \tag{16}$$

$$x^3 + y^2 = 1 \tag{17}$$

and denoting the *tetrahedron-irrationality* by ξ,

$$\xi = \frac{x^3-4}{-3x^2}, \quad \sqrt{(\xi^3-1)} = \frac{(y^2-9)y}{3^{\frac{3}{2}}(y^2-1)} \tag{18}$$

$$x = -\tau_{18} - 2, \quad y = \tau_{12} - 3 \tag{19}$$

$$2\tau_6 + 8 = (\tau_{18}+2)^3 = -x^3, \quad 2\tau_6 + 9 = (\tau_{12}-3)^2 = y^2 \tag{20}$$

and GIERSTER's τ_{12} ('Math. Ann.,' 14) is given by

$$\tau_{12} = \frac{36}{y^2-9} = \frac{36}{-x^3-8} \tag{21}$$

So also $\mu = 48$ can be discussed by a trisection of $\mu = 16$, by putting

$$(\tfrac{1}{o}+o)^2 = \tfrac{1}{4}\left(\frac{1}{a}-a\right)^4 = \frac{(b-1)^3(b+3)}{4b} \tag{22}$$

49. $\mu = 32,\ v = \mathrm{K} + f\mathrm{K}'i,\ f = \frac{1, 3, 5, 7, 9, 11, 13, 15}{16}.$

$$[s(\omega) - s(8v)]^2 = \frac{m^4\alpha^4 a^4}{(\alpha - m)^4} \tag{1},$$

and this leads to

$$\beta^3 - (1 + 2m - 2m^2)\beta^2 + 4(m - m^2)\beta - (m - m^2)$$
$$- \beta(\beta - 1)(\beta - 1 + 2m - 2m^2)a = 0 \tag{2},$$

$$m - m^2 = \frac{\beta(\beta - 1)[\beta(1 - a) + a]}{2\beta^2(1 + a) - 2\beta(2 + a) + 1}$$
$$= \frac{b(1 + ab)}{(1 + b)(1 + 2b - b^2 + 2ab)} \tag{3},$$

so that

$$(1 + ab)\{(1 + 2b)^2 - a^4\}$$
$$- (1 + b)^2(1 + 2b - b^2 + 2ab) = 0 \tag{4},$$

$$b^4 + 2ab^3 - a(a^4 + 1)b - a^4 = 0 \tag{5},$$

a C_6 in (a, b).

Put $b = ac$,

$$a^4c - (c^4 + 2c^3 + 0 + 0 - 1)a^2 + c = 0 \tag{6},$$

a quadratic in a^2,

$$a^2 + \frac{1}{a^2} = \frac{c^4 + 2c^3 + 0 + 0 - 1}{c} \tag{7},$$

$$\left(a + \frac{1}{a}\right)^2 = \frac{(c^2 + 1)(c^2 + 2c - 1)}{c} \tag{8},$$

$$\left(a - \frac{1}{a}\right)^2 = \frac{(c + 1)^3(c - 1)}{c} \tag{9},$$

$$a = \frac{\sqrt{c^2 + 1}\,.\,c^2 + 2c - 1 + (c + 1)\sqrt{c^2 - 1}}{2\sqrt{c}} \tag{10},$$

$$b = \frac{\sqrt{c^2 + 1}\,.\,c^2 + 2c - 1 + (c + 1)\sqrt{c^2 - 1}}{2}\sqrt{c} \tag{11},$$

$$\mathrm{C}^2 = \tfrac{1}{4}\left(\frac{1}{a} - a\right)^2\left[\frac{1}{c^2}\left(\frac{1}{a} + a\right)^2 - 4\right]$$
$$= \frac{(c + 1)^3(c - 1)}{4c}\left[\frac{(c^2 + 1)(c^2 + 2c - 1)}{c^3} - 4\right]$$
$$= \frac{(c^2 - 1)^4}{4c^4} \tag{12},$$

$$\mathrm{C} = \tfrac{1}{o} + o = \tfrac{1}{2}\left(c - \frac{1}{c}\right)^2 \tag{13},$$

$$\frac{1}{\sqrt{o}} + \sqrt{o} = \frac{1}{\sqrt{2}}\left(c + \frac{1}{c}\right) \qquad (14),$$

as in $\mu = 16$, so that the bisection formulas can be carried one stage further.

We find now

$$\frac{\operatorname{dn} \frac{1}{16} \mathrm{K}'}{o} = \frac{\sqrt{\left[1 + \frac{2c^{\frac{3}{2}}}{\sqrt{(c^2+1)}\,(c^2+2c-1)}\right]} - \sqrt{\left[1 - \frac{2c^{\frac{3}{2}}}{(c+1)\sqrt{c^2-1}}\right]}}{\sqrt{\left[1 + \quad ,, \quad\right]} + \sqrt{\left[1 - \quad ,, \quad\right]}} \qquad (15).$$

50. $\mu = 40$, $v = \mathrm{K} + f\mathrm{K}'i$, $f = \frac{2r+1}{20}$, $= \frac{1,\,3,\,7,\,9,\,11,\,13,\,17,\,19}{20}$.

The relation

$$[s(\omega) - s(10v)]^2 = (s_1 - s_3)(s_2 - s_3) \qquad (1)$$

leads to

$$\mathrm{N}_{10} + a\mathrm{D}_{10} = 0 \qquad (2),$$

or putting $m - m^2 = n$,

$$(1-4n)\beta^4 - 2(2-7n)\beta^3 + 2(3-11n+2n^2)\beta^2 - (4-17n+10n^2)\beta + 1 - 5n + 5n^2$$
$$+ a(\beta^2 - \beta + n)[(1-4n)\beta^2 - (1-bn)\beta - n] = 0 \qquad (3).$$

With $\beta = \frac{1}{b+1}$, and arranged in powers of n,

$$(b+1)^2[5b^2 - 1 - a(b^2 - 4b - 1)]n^2$$
$$- [5b^3 + 3b^2 + b - 1 + 2a(2b+1)]bn$$
$$+ b^2(b^2 + a) = 0 \qquad (4),$$

and with

$$\frac{b}{n} = \frac{(1+2b)^2 - a^4}{(1+b)} \qquad (5)$$

this becomes

$$(b+1)^4[5b^2 - 1 - a(b^2 - 4b - 1)]$$
$$- (b+1)[5b^3 + 3b^2 + b - 1 + 2a(3b+1)][(2b+1)^2 - a^4]$$
$$+ (b^2 + a)[(2b+1)^2 - a^4]^2 = 0 \qquad (6).$$

A factor $1 - a$ may be cancelled and

$$b^6 + 0\,.\,b^5 + \mathrm{A}_2 b^4 + \mathrm{A}_3 b^3 + \mathrm{A}_4 b^2 + 0b$$
$$- a^4(1 + a + a^2 + a^3 + a^4) = 0 \qquad (7).$$

$$\mathrm{A}_2 = 3a(1 + a + a^2), \quad \mathrm{A}_3 = 0, \quad \mathrm{A}_4 = -a(1 + a + a^2 - a^3 + a^4 + a^5 + a^6) \qquad (8).$$

Put $b = ac$,

$$c^6 + 3\left(\frac{1}{a} + 1 + a\right)c^4 - \left(\frac{1}{a^3} + \frac{1}{a^2} + \frac{1}{a} - 1 + a + a^2 + a^3\right)c^2 - \left(\frac{1}{a^2} + \frac{1}{a} + 1 + a + a^2\right) = 0 \qquad (9).$$

Put $\dfrac{1}{a} + a = t$,

$$c^6 + 3(t+1)c^4 - (t^3 + t^2 - 2t - 3)c^2 - t^2 - t + 1 = 0 \qquad (10).$$

Put $c^2 = x - 1$,

$$x^3 + 3tx^2 - t(t^2 + t + 4)x + t^3 = 0 \qquad (11).$$

Put $x = yt$,

$$ty^3 + 3ty^2 - (t^2 + t + 4)y + t = 0 \qquad (12).$$

Put

$$y = \frac{1-z}{1+z} \qquad (13),$$

$$t = 2\frac{(\sqrt{Z}+1)^2}{(1-z)(1+z)^2}, \quad Z = -z + z^2 + z^3 \qquad (14),$$

$$x = 2\frac{(\sqrt{Z}+1)^2}{(1+z)^3} \qquad (15),$$

$$c^2 = \frac{1 - 5z - z^2 + z^3 + 4\sqrt{Z}}{(1+z)^3} \qquad (16),$$

$$t - 2 = \frac{4\sqrt{Z}(\sqrt{Z}+1)}{(1-z)(1+z)^2} \qquad (17),$$

$$t + 2 = \frac{4(\sqrt{Z}+1)}{(1-z)(1+z)^2} \qquad (18),$$

$$t^2 - 4 = \frac{16\sqrt{Z}(\sqrt{Z}+1)^2}{(1-z)^2(1+z)^4} = \left(\frac{1}{a} - a\right)^2 \qquad (19).$$

$$C^2 = 64z^2\sqrt{Z}\,\frac{(1+z^2)(1+2z-6z^2-2z^3-z^4) - 8z^2\sqrt{Z}}{(1-z^2)^5(1+4z-z^2)}$$

$$= \frac{64z^2\sqrt{Z}}{(1+z^2)(1+2z-6z^2-2z^3+z^4) + 8z^2\sqrt{Z}} \qquad (20),$$

as in (2), § 42, for $\mu = 20$.

To connect up the results, $\mu = 20$ and 40,

$$\frac{s(\omega) - s(10v)}{s(\omega) - s(2v)} = a^2 \text{ in } \mu = 40 \tag{21}$$

$$\frac{s(\omega) - s(5v)}{s(\omega) - s(v)} = \left(\frac{\epsilon + p}{\epsilon - p}\right)^2 \text{ in } \mu = 20 \tag{22}$$

are equal, so that

$$a_{40} = \frac{p + \epsilon}{p - \epsilon} = \frac{a_{20} + b_{20}}{a_{20} - b_{20}} \tag{23}$$

and thus, with $a_{20} = a$, $b_{20} = b$,

$$\frac{a^2 + b^2}{a^2 - b^2} = \tfrac{1}{2}t = \frac{1 - z + z^2 + z^3 + 2z\sqrt{Z}}{(1 - z)(1 + z)^2} \tag{24}$$

$$\frac{b^2}{a^2} = \sqrt{Z} \tag{25}$$

$$a_{20} = \frac{1}{z_{40}} \tag{26}$$

(KIEPERT, 'Math. Ann.,' 32, p. 119), and the preceding results are thus merely bisection formulas for $\mu = 20$.

We arrive at the conclusion that it is the quotient of two theta functions, θu and $\theta(u - v)$, with constant phase difference v, which is required in dynamical application, the functions α, β, γ, δ for instance employed by KLEIN in top-motion; but the separate theta function θu has no mechanical interest.

This quotient, qualified by the constant factors $\theta 0$ and θv, is an elliptic function of u when v is a half-period, dn u for instance when v is the half-period K, and the quotient is the μ^{th} root of an algebraical function of the elliptic functions of u when v is an aliquot μ^{th} part of a period; in this way we express the result of ABEL's pseudo-elliptic integral.

The formation of this algebraical function for the simplest values of μ has been our chief object, and in the course of the work the ellipsotomic problem has been carried out of the determination of the Division-Values of the Elliptic Function.

The Transformation problem may be considered solved at the same time by means of symmetric functions of the division-values; but as Transformation has no dynamical utility, it has not been developed in this memoir.